Consider a variant of our standard model in which velocity can be greater than 1. Assume that the cash-in-advance constraint is given by

$$M_t \geq \kappa P_t C_t,$$

where κ captures the extent to which cash is used in transactions. The remaining portion $(1-\kappa)P_tC_t$ must be paid for in the asset market at the end of the period. As a result, our asset market budget constraint is unchanged, and still given by

$$P_t Z_t L_t + M_t - P_t C_t + B_t + T_t - M_{t+1} - q_t B_{t+1} = 0.$$

All the other aspects of our model are standard, including the household's payoff which is given by

$$\sum_{t=1}^{T} \beta^{t-1} \left[u(C_t) - v(L_t)\right] + \beta^T V(M_{T+1}, B_{T+1}).$$

The market clearing conditions are also unchanged and given by

$$\begin{aligned} C_t &= Z_t L_t \\ B_{t+1} &= 0 \\ M_{t+1} &= \bar{M}_{t+1} \end{aligned}$$

where the aggregate money supply grows according to $\bar{M}_{t+1} = (1+\tau)\bar{M}_t$ and transfers are therefore given by $T_t = \tau \bar{M}_t$. Assume also that productivity grows according to $Z_{t+1} = (1+g)Z_t$. Finally assume that our preferences take the standard forms

$$u(C_t) = log(C_t), \quad v(L_t) = \frac{L_t^{1+\gamma}}{1+\gamma}.$$

A) Write down the Lagrangian for this problem and determine the first-order conditions for the optimal levels of consumption, labor, money and bonds.

B) If we assume that the cash-in-advance constraint binds and that the goods market clearing condition holds, explain how this pins down the equilibrium price of consumption P_t. If labor L_t was constant (given κ) what would this lead you to predict about the rate of change in the price level P_{t+1}/P_t?

Use your result here to discuss how lowering κ from our standard value of 1 can affect the relationship in a standard empirical velocity equation (which we use to measure v):

$$Mv = PY.$$

C) We want to reduce our equilibrium conditions down to a simple 3 equation system for $L_t = L$ and our two multipliers. Explain why we will need to make a change-in-variables with respect to our multipliers, make the appropriate change and derive the three key equations. Also derive an expression for q_t in the steady state.

D) We previously saw (in the model with $\kappa = 1$) that the solution to the social planner's problem boiled down to simply solving for L such that it maximized

$$log(Z_t L_t) - v(L_t)$$

(Note that the optimal value of L does not depend upon Z_t.) Denote this optimal level of labor by L^* and solve for it given our functional forms for u and v. If we guess that this is still the best we can do, does this still require that $q_t = 1$? And, if so, why?

E) Assume that you were able to solve the fundamental 3-equation system in part (C). Assume that this yielded a labor function $L(\tau, \kappa)$. How would you expect L to depend upon these two variables? Also fixing τ, what does reducing κ do to the gap between $L^* - L(\tau, \kappa)$? What does this say about the cost of inflation?