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Advanced Macroeconomics
Second Exam 2024/25
Part 1

First, write your name on your exam. **Second,** this part of the exam has two questions. The second question has multiple sub-questions. The sub-questions build on each other, so it is important to take your time and get a sub-question right before moving on to the next one. **Third,** please be sure to use your time wisely and show your work. Good luck.

1 [6pts.]. Consider a variant of the model of capital covered in class where one must also use money to acquire capital. Because of this, the cash-in-advance constraint is now given by

$$M_t \geq P_t[C_t + X_t].$$

To make that analytics easier, assume that depreciation is 100% (i.e. $\delta = 1$), so $X_t = K_{t+1}$. The model is otherwise unchanged. Assume the utility function:

$$\log(C_t) - \frac{L_t^{1+\gamma}}{1+\gamma}.$$

Characterize a balanced growth equilibrium, being careful to explain your work. Derive the analogs of the fundamental system of equations and discuss how they have changed and the implications of those changes:

The equations from the original version of the model are reproduced below. The first is the period 1 resource constraint:

$$[Z_1 L]^{1-\alpha} K_1^{\alpha} = C_1 + [g+\delta] K_1.$$

The second is our optimal labor condition,

$$L^{\gamma+\alpha} = \frac{\beta}{1+\tau} P_1 (1-\alpha) Z_1^{1-\alpha} K_1^{\alpha}. \tag{1}$$

The third is our optimal capital condition,

$$1 = \frac{\beta}{1+g}\left\{\alpha\left[\frac{Z_1 L}{K_1}\right]^{1-\alpha} + 1 - \delta\right\}. \tag{2}$$

The fourth is our optimal consumption condition,

$$1 = \left[\tilde{\lambda} + \tilde{\mu}\right].$$

The fifth is our condition for money which implies that

$$\tilde{\mu} = \frac{\beta}{1+\tau}.$$

The sixth is our price level condition

$$P_t = \frac{M_t}{C_t}.$$

Do not forget to provide the interpretation for the new capital condition.

2. Consider a version of our New Keynesian model. We will assume that the household's shopper receives the monetary injection *after* entering the goods market and the household's seller does not know its size when they make their pricing decision. The demand for an individual's good is given by

$$D\left(\frac{P_t(k)}{\bar{P}_t}\right)$$

where $P_t(k)$ is his/her price and $\bar{P}_t$ is the price index. Note that

$$\frac{d}{dP(k)} D\left(\frac{P(k)}{\bar{P}}\right) = D\left(\frac{P(k)}{\bar{P}}\right) \frac{1}{\rho - 1} P(k)^{-1}.$$

The maximization problem is given by

$$\max_{\{C_t, P_t(i), M_{t+1}, B_{t+1}\}_{t=1,2}} \mathbb{E}\left\{ \begin{array}{c} u(C_1) - v\left(D\left(\frac{P_1(i)}{\bar{P}_1}, \tau_1\right)/Z_1\right) + \beta\left[u(C_2) - v\left(D\left(\frac{P_2(i)}{\bar{P}_2}, \tau_2\right)/Z_1\right)\right] \\ +\beta^2 V(M_3, B_3) \end{array} \right\}$$

subject to

$$\begin{aligned} M_t + T_t &\geq \bar{P}_t C_t \text{ and} \\ P_t(i) D\left(\frac{P_t(i)}{\bar{P}_t}\right) + \left[M_t + T_t - \bar{P}_t C_t\right] + B_t &\geq M_{t+1} + q_t B_{t+1} \text{ for } t = 1, 2. \end{aligned}$$

A)[2pts.] Derive the first-order condition for setting the price $P_t(i)$. Then, using this first-order condition, making use of the fact that $P_t(i) = \bar{P}_t$ since all prices are the same in equilibrium, and that $D\left(\frac{P_t(i)}{\bar{P}_t}, \tau_t\right) = Z_t L_t$, show that we get that

$$0 = \mathbb{E}_t\left\{-\beta^{t-1} v'(L_t) L_t + \mu_t \rho Z_t L_t \bar{P}_t\right\}.$$

B)[2pts.] Derive the first-order conditions for consumption and money. Remember that these decisions are taken after all uncertainty of period t is resolved and hence we get our standard conditions. Assume that preferences over consumption are log, and the cash-in-advance constraint holds, so

$$\bar{P}_t = \frac{M_t(1+\tau_t)}{Z_t L_t},$$

Show that you can use all this to derive the following

$$0 = \mathbb{E}_t\left\{-v'(L_t) L_t + \beta \frac{1}{(1+\tau_{t+1})} \rho\right\} \tag{3}$$

To finish deriving an expectational Phillips curve, remember that the realized level of labor, L_t bears the following relationship to the target level of labor $\bar{L}_t$,

$$L_t = \frac{(1+\tau_t)}{(1+\bar{\tau}_t)}\bar{L}_t. \tag{4}$$

where $\bar{\tau}_t$ is the expected level of money growth at date t. If we assume that $v(L_t) = \frac{L_t^{1+\gamma}}{1+\gamma}$ and plug the target level of labor into equation (3), we get an expression for how this target will respond to the expected growth rate of money. As an approximation this can be taken to be

$$\bar{L}_t^{1+\gamma} = \frac{\beta\rho}{1+\bar{\tau}_{t+1}}. \tag{5}$$

C)[2pts.] Consider a country that is running a hyperinflation, so $\bar{\tau}_t$ is very large. This country then slows the growth rate of money abruptly. Discuss what you would expect to have occur under the following scenarios:

1. Expectations gradually adjust down to the new low level because the country has heard many claims about fighting inflation before.

2. Expectations adjust immediately because the current leader is known to be phobic about inflation.

Please use in your discussion the conditions, (3) and (4).

In a very interesting study, Thomas Sargent pointed out that at the ends of the four major hyperinflations that he studied, output and real money demand recovered rapidly. Under which of these two scenarios would we see this sort of outcome.

Here is the answer to **1**

Assume perfect foresight so HH's problem is choosing sequence $\{C_t, L_t, M_{t+1}, B_{t+1}, K_{t+1}\}_{t=1}^{2}$ so as to maximize

$$\max \sum_{t=1,2} \beta^{t-1} \left[u(C_t) - v(L_t)\right] + \beta^2 V(M_3, B_3, K_3)$$

subject to

$$M_t \geq P_t[C_t + K_{t+1}] \text{ and}$$

$$\begin{aligned} & P_t \left[Z_t L_t\right]^{1-\alpha} K_t^{\alpha} + \left[M_t - P_t C_t\right] + B_t + T_t \\ \geq \ & M_{t+1} + q_t B_{t+1} + P_t K_{t+1} \text{ for all } t \leq 2. \end{aligned}$$

The household's consumption condition is unchanged

$$\beta^{t-1} u'(C_t) - P_t \left[\lambda_t + \mu_t\right] = 0.$$

The f.o.c. for L_t is unchanged:

$$-\beta^{t-1} v'(L_t) + \mu_t P_t Z_t^{1-\alpha} (1-\alpha) L_t^{-\alpha} K_t{}^{\alpha} = 0.$$

The f.o.c.s for money and bonds are also unchanged and are given by

$$-\mu_t + \lambda_{t+1} + \mu_{t+1} = 0$$

and

$$-\mu_t q_t + \mu_{t+1} = 0.$$

The condition for choice of capital stock at the end of period has changed

$$-[\mu_t + \lambda_t] P_t + \mu_{t+1} P_{t+1} \left[Z_{t+1} L_{t+1}\right]^{1-\alpha} \alpha K_{t+1}^{\alpha-1} = 0.$$

We are assuming that both productivity and the money supply grow at constant rates and normalize initial values to get

$$Z_t = (1+g)^{t-1},$$

and

$$M_t = (1+\tau)^{t-1}$$

Conjecture that capital grows at the same rate as productivity and that labor is constant, which implies that output will grow at rate g since

$$Y_t = \left[Z_t L\right]^{1-\alpha} K_t^{\alpha} \implies Y_t = Z_t Y_1$$

given our normalization of $Z_1 = 1$.

We will also conjecture that the cash-in-advance constraint holds as an equality. So,

$$P_t = \frac{M_t}{C_t + X_t} = \frac{M_t}{Y_t} = \frac{(1+\tau)}{(1+g)} P_{t-1}.$$

Note that this condition has changed.

Assuming log utility, our f.o.c. for consumption is

$$\beta^{t-1}\frac{1}{C_t} = [\lambda_t + \mu_t]\, P_t = [\lambda_t + \mu_t]\, \frac{M_t}{Y_t}$$

So once again, we make a change in variables, setting

$$\tilde{\lambda}_t = \lambda_t \frac{M_t}{\beta^{t-1}},$$

$$\tilde{\mu}_t = \mu_t \frac{M_t}{\beta^{t-1}}.$$

This gives us

$$\frac{Y_1}{C_1} = \left[\tilde{\lambda}_t + \tilde{\mu}_t\right]$$

since both output and consumption grow at the same rate. This also indicates that the sum of our multipliers is invariant, suggesting that again they both are.

The f.o.c. for labor can be written as follows once we substitute for P_t and μ_t

$$\begin{aligned}
\beta^{t-1} L_t^{\gamma} &= \tilde{\mu}\frac{\beta^{t-1}}{M_t}\frac{M_t}{Y_t} Z_t^{1-\alpha}(1-\alpha)L_t^{-\alpha}K_t^{\alpha} \\
L_t^{\gamma} &= \tilde{\mu}\frac{1}{Y_t} Z_t^{1-\alpha}(1-\alpha)L_t^{-\alpha}K_t^{\alpha} = \tilde{\mu}\frac{1}{Y_1}(1-\alpha)\left(\frac{K_1}{L_t}\right)^{\alpha}
\end{aligned}$$

which indicates that labor is going to be constant as assumed.

The f.o.c.'s for money can be written as

$$\tilde{\mu}_t = \frac{\beta}{1+\tau}\left[\tilde{\lambda}_{t+1} + \tilde{\mu}_{t+1}\right] = \frac{\beta}{1+\tau}\frac{Y_1}{C_1}$$

which is slightly different from what we had before. The f.o.c. for bonds still leads to

$$\tilde{\mu}_t q_t = \frac{\beta}{1+\tau}\tilde{\mu}_{t+1} \implies q_t = \frac{\beta}{1+\tau}.$$

which is exactly what we had before.

Making use of our result for $\tilde{\mu}$, our labor equation becomes

$$L_t^{\gamma} = \frac{\beta}{1+\tau}\frac{Y_1}{C_1}\frac{1}{Y_1}(1-\alpha)\left(\frac{K_1}{L_t}\right)^{\alpha} = \frac{\beta}{1+\tau}\frac{1}{C_1}(1-\alpha)\left(\frac{K_1}{L_t}\right)^{\alpha}$$

Note here that $1/C_1$ is exactly our old P_1, so the labor equation is effectively the same as before. However, here this does not correspond to $1/P_1$ as in the old model.

The f.o.c. for capital,

$$-[\mu_t + \lambda_t]P_t + \mu_{t+1}P_{t+1}\left[Z_{t+1}L_{t+1}\right]^{1-\alpha}\alpha K_{t+1}^{\alpha-1} = 0.$$

can be rewritten as follows once we substitute out for prices and make our change-in-variables

$$
\begin{aligned}
\tilde{\mu} + \tilde{\lambda} &= \frac{\beta}{1+\tau}\tilde{\mu}\frac{Y_1}{C_1}\frac{1+\tau}{1+g}\alpha\left[\frac{Z_{t+1}L_{t+1}}{K_{t+1}}\right]^{1-\alpha} \\
\Longrightarrow \frac{Y_1}{C_1} &= \frac{\beta}{1+\tau}\frac{\beta}{1+\tau}\frac{Y_1}{C_1}\frac{1+\tau}{1+g}\alpha\left[\frac{Z_{t+1}L_{t+1}}{K_{t+1}}\right]^{1-\alpha} \\
\Longrightarrow 1 &= \frac{\beta}{1+\tau}\frac{\beta}{1+g}\alpha\left[\frac{Z_1 L}{K_1}\right]^{1-\alpha},
\end{aligned}
$$

where, we used our prior results for the stationary multipliers, and also the fact that Z_t and K_t grow at the same rate and labor is constant.

This implies a lower level of the capital-labor ratio because of the extra $\beta/(1+\tau)$ term relative to one in which the HH did not have to use money to buy capital. Also, the price level will be lower conditional on K and L because all of output is now being bought with money.

The expression also has a nice interpretation. There are now two delays associated with investing. The first is between the period in which the wealth is earned, and the investment is made. This leads to the same term as in the labor equation and for the same reason, wealth must be carried over in the form of money for one period . The second term is the standard capital delay condition coming from waiting one period and having output be higher by the factor $1+g$.

Here is the answer to **2**

A.

The HH objective is given by

$$max_{P_1(i)}\mathbb{E}_1\Bigg\{max_{C_1,M_1,B_1}u(C_1)-v\left(D\left(\frac{P_1(i)}{\bar{P}_1},\tau_1\right)/Z_1\right)$$
$$+\beta max_{P_2(i)}\mathbb{E}_2\Bigg\{max_{C_2,M_2,B_2}u(C_2)-v\left(D\left(\frac{P_2(i)}{\bar{P}_2},\tau_2\right)/Z_2\right)+...\Bigg\}\Bigg\}+...$$

In a period, the HH first chooses its price $P_t(i)$ knowing τ_{t-1} and being able to infer $\bar{P}_t$ hence $\mathbb{E}_1$. Then it finds out τ_t and chooses $\{C_t, M_{t+1}, B_{t+1}\}$ knowing also q_t

Considering only the parts that involve P_1 in the Lagrangian and ignoring the others

$$\mathbf{L} = max_{P_1(i)}\mathbb{E}_1\Bigg\{-v\left(D\left(\frac{P_1(i)}{\bar{P}_1},\tau_1\right)/Z_1\right)+\mu_1(\tau_1)P_1(i)D\left(\frac{P_1(i)}{\bar{P}_1},\tau_1\right)\Bigg\}$$

The first-order condition for the price will be

$$\sum_{\tau_1}\Bigg\{-v'\left(D\left(\frac{P_1(i)}{\bar{P}_1},\tau_1\right)/Z_1\right)\frac{1}{Z_1}\frac{\partial}{\partial P_1(i)}D\left(\frac{P_1(i)}{\bar{P}_1},\tau_1\right)$$
$$+\mu_1(\tau_1)D\left(\frac{P_1(i)}{\bar{P}_1},\tau_1\right)+\mu_1(\tau_1)P_1\frac{\partial}{\partial P_1(i)}D\left(\frac{P_1(i)}{\bar{P}_1},\tau_1\right)\Bigg\}Pr\{\tau_1\}=0$$

Using the expression for the derivative we get:

$$\sum_{\tau_1}\Bigg\{-v'\left(D\left(\frac{P_1(i)}{\bar{P}_1},\tau_1\right)/Z_1\right)\frac{1}{Z_1}D\left(\frac{P_1(i)}{\bar{P}_1}\right)\frac{1}{\rho-1}P_1(i)^{-1}$$
$$+\mu_1(\tau_1)D\left(\frac{P_1(i)}{\bar{P}_1},\tau_1\right)+\mu_1(\tau_1)P_1(i)D\left(\frac{P_1(i)}{\bar{P}_1}\right)\frac{1}{\rho-1}P_1(i)^{-1}\Bigg\}Pr\{\tau_1\}=0$$

Then, again making use of the fact that (i) $P_t(i) = \bar{P}_t$ since all prices are the same in equilibrium, and (ii) that $D\left(\frac{P_t(i)}{\bar{P}_t},\tau_t\right) = Z_tL_t$, to get that

$$E_t\Bigg\{-\beta^{t-1}v'(L_t)L_t\frac{1}{\rho-1}\bar{P}_t^{-1}+\mu_t(\tau_t)Z_tL_t+\mu_t(\tau_t)Z_tL_t\frac{1}{\rho-1}\Bigg\}=0$$

$$\mathbb{E}_t(-\beta^{t-1}v'(L_t)L_t+\mu_t\rho Z_tL_t\bar{P}_t)=0$$

B.

The first order condition for consumption is:

$$\beta^{t-1}u'(C_t) - \bar{P}_t\left[\lambda_t + \mu_t\right] = 0.$$

and the first order condition for money is:

$$-\mu_t + E_t(\lambda_{t+1} + \mu_{t+1}) = 0$$

Replacing these conditions in

$$\mathbb{E}_t(-\beta^{t-1}v'\left(L_t\right)L_t + \mu_t\rho Z_t L_t \bar{P}_t) = 0$$

we get

$$\mathbb{E}_t(-\beta^{t-1}v'\left(L_t\right)L_t + (\lambda_{t+1} + \mu_{t+1})\rho Z_t L_t \bar{P}_t) = 0$$

and

$$\mathbb{E}_t\Big\{-v'\left(L_t\right)L_t + \beta\frac{u'(C_{t+1})}{\bar{P}_{t+1}}\rho Z_t L_t \bar{P}_t\Big\} = 0$$

Assuming preferences on consumption are log and the CIA holds with equality we have:

$$\frac{u'(C_{t+1})}{\bar{P}_{t+1}} = \frac{1}{M_{t+2}}.$$

Replacing this in the equation with the expectation operator we get

$$0 = \mathbb{E}_t\left\{-v'\left(L_t\right)L_t + \beta\frac{1}{(1+\tau_{t+1})}\rho\right\}.$$

Assuming certainty equivalence and assuming that $\bar{L}_t$ is the expected level of labor we can write (3) as

$$0 = -v'\left(\bar{L}_t\right)\bar{L}_t + \beta\frac{1}{(1+\bar{\tau}_{t+1})}\rho.$$

Using the functional form for v we get equation (5)

$$\bar{L}_t = \left[\frac{\beta\rho}{1+\bar{\tau}_{t+1}}\right]^{1/(1+\gamma)}$$

C. Consider equations (4) and (5) :

$$L_t = \frac{(1+\tau_t)}{(1+\bar{\tau}_t)}\bar{L}_t,$$

and

$$\bar{L}_t = \left[\frac{\beta \rho}{1 + \bar{\tau}_{t+1}} \right]^{1/(1+\gamma)}. \tag{6}$$

If the country leader follows a simple rule for the money growth

$$\tau_t = \tilde{\tau}_t + \varepsilon_t.$$

The citizens do not know what rule has been followed, and only observe the previous money growth. Using these series they compute the historical expected money growth rate of τ_t, which we denote by $\bar{\tau}_t$. The $\bar{\tau}_t$ has been very large until date t^*, but the country leader decides to have a much smaller τ_t for $t > t^*$. Under scenario 1, the leader is not credible and $\bar{\tau}_t$ drops slowly until it reaches the true expected money growth rate. According to equations (4) the expected employment $\bar{L}_t$ (and production) will continue to be low for a long period since $\bar{\tau}_{t+1}$ is expected to continue to be high. According to equation (5) the actual employment (and production) will be lower than the expected employment because $\frac{(1+\tau_t)}{(1+\bar{\tau}_t)} < 1$. Overtime L_t increases as $\bar{\tau}_t$ approaches τ_t. Under scenario 2 $\bar{\tau}_{t+1}$ will adjust immediately. So according to (4) $\bar{L}_t$ (and expected production) will increase substantially and immediately. According to equation (5) the actual employment (and production) will be similar to the expected employment because $\frac{(1+\tau_t)}{(1+\bar{\tau}_t)}$ will be a value in the neighborhood of 1 as long as the shock ε_t is small.

According to the model, if the citizens believe the countries' leaders are credible (scenario 2), real money demand and output can recover quickly. Otherwise (scenario 1), the disinflationary process can be slow and costly.