Universidade de Lisboa
Instituto Superior de Economia e Gestão
BsC in Economics, Finance and Management

Mathematics II – 1st Semester - 2022/2023

Resit Assessment - 09th of January 2025

Duration: $(120+\varepsilon)$ minutes, $|\varepsilon| \leq 30$

Version A

Name: ....

Student ID #: ....

Part I

- Complete the following sentences in order to obtain true propositions. The items are independent from each other.
- There is no need to justify your answers.

(a) *(8)* If $A:\mathbb{R}^3 \to \mathbb{R}^3$ is a linear map such that

$$A(1,0,0)=(5,0,0), \quad A(0,1,0)=(0,-1,0) \quad \text{and} \quad A(0,0,2)=(0,0,10),$$

then the eigenvalues of A^{-1} are .... and their algebraic multiplicities are ...., respectively.

(b) *(3)* The quadratic form Q associated to the matrix $\begin{pmatrix} 0 & -3 \\ -3 & 2 \end{pmatrix}$ is given by

$$Q(x,y) = \text{...}$$

(c) *(5)* Consider the map $f: D_f \to \mathbb{R}$ defined by

$$f(x,y) = \frac{1}{y-1} + \ln(x^2 y)$$

The set D_f may be explicitly written as:

$$D_f = \text{...}$$

(simplify the expression)

(d) *(6)* The set of accumulation points of $\left\{\left(\sin(\frac{n\pi}{2}), \left(\frac{2}{3}\right)^n\right), n \in \mathbb{N}\right\} \subset \mathbb{R}^2$ is

....

(e) *(6)* Consider the set $A = \{(x, y) \in \mathbb{R}^2 : x^2 + (y+2)^2 < 4\}$.
The border (frontier) of A is analytically defined by

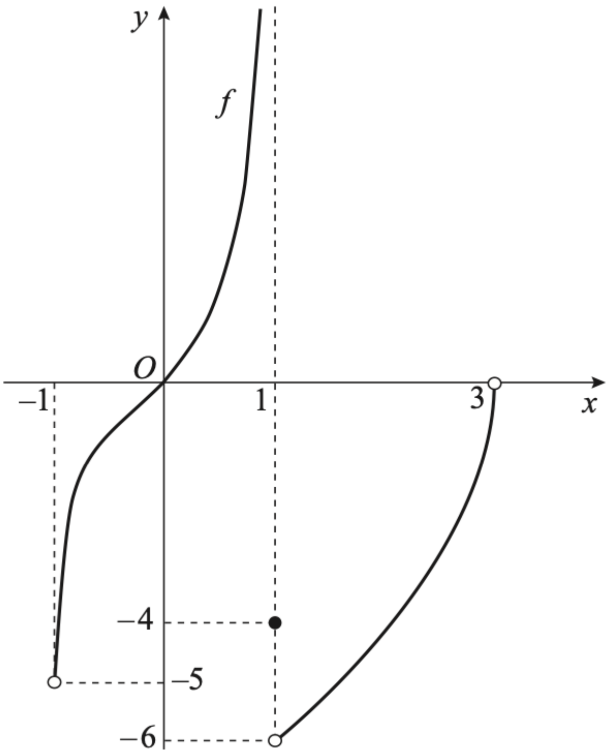
$\partial A =$ ....

Since $cl(A) \neq A$, then A is not .... and, consequently, A is not compact.

(f) *(5)* The graph of $f :]-1, 3[\to \mathbb{R}$ is below. We know that

$$\lim_{x \to 1^-} f(x) = +\infty, \qquad f(1) = -4$$

and $(x_n)_{n\in\mathbb{N}}$ is a sequence in $]1, 3[$ such that $\lim_{n\in\mathbb{N}} x_n = 1$.



Then

$$\lim_{n \to +\infty} f(x_n) = \text{.........}$$

.

(g) *(6)* The continuous map $f(x,y) = e^{-\frac{1}{2}(x^2+y^2)}$ does not have a global minimum when restricted to $M = \{(x,y) \in \mathbb{R}^2 : y = 0\}$. This does not contradict ....'s Theorem because M is not .... and, consequently, it is not compact.

(h) *(6)* With respect to the continuous map $f : \mathbb{R}^2 \to \mathbb{R}$, its directional derivatives along the non-null vector (v_1, v_2) at $(1,2)$ are given by

$$D_{(v_1,v_2)}f(1,2) = \frac{(v_1+2)^2}{1+v_2^2}.$$

Hence we may conclude that:

$$\frac{\partial f}{\partial x}(1,2) = \ldots\ldots\ldots\ldots \quad \text{and} \quad \frac{\partial f}{\partial y}(1,2) = \ldots\ldots\ldots\ldots$$

(i) *(6)* We know that $g : \mathbb{R}^2 \to \mathbb{R}$ is a C^2 map. If $f : \mathbb{R}^2 \to \mathbb{R}$ and $u, v : \mathbb{R} \to \mathbb{R}$ are such that

$$f(x,y) = xy.g(u,v), \quad u(x) = x^2 \quad \text{and} \quad v(x) = 4x^2,$$

then (using the Chain rule):

$$\frac{\partial f}{\partial y}(x,y) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

(j) *(5)* With respect to a smooth map $f : \mathbb{R}^2 \to \mathbb{R}$, one knows that

$$\nabla f(x,y) = (4 - 2x\sin(x^2+y); -\sin(x^2+y)).$$

If $f(x,y)$ does not have constant terms, then

$$f(x,y) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

(k) *(4)* With respect to the map $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = 2x - y - x^2 - y^2$, one knows that $\nabla f(1, -1/2) = (0,0)$. Since

$$H_f(x,y) = \begin{pmatrix} \ldots\ldots & 0 \\ \ldots\ldots & \ldots\ldots \end{pmatrix},$$

then f is stricly concave in $\mathbb{R}^2$ and thus $f(1,-1/2)$ is a .... maximum of f.

(l) *(6)* The Taylor polynomial of order 2 of the map $f(x,y) = e^{-3y}$ at the point $(0,0)$ is given by

$$P_{f,2,(0,0)}(h_1,h_2) = \text{..}$$

(m) *(3)* The minimisers and maximisers of the map $f : \mathbb{R}^3 \to \mathbb{R}$ given by

$$f(x,y,z) = x - 2y + 2z$$

restricted to the set

$$M = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

are critical points of the map:

$$\mathcal{L}(x,y,z,\lambda) = \text{..}$$

(n) *(5)* The **area** of the semi-circle defined in $\mathbb{R}^2$ by $x^2 + y^2 \leq 4 \;\wedge\; y \geq 0$ may be computed using the following double integral:

$$\int_{....}^{.....} \int_{0}^{.......................} \text{.....}\ \mathrm{dy}\ \mathrm{dx} = 2\pi$$

(o) *(6)* The map $y(x) = (1+x)e^{-2x}$, $x \in \mathbb{R}$, is a solution of the IVP

$$\begin{cases} y'' + \text{......}y' + \text{......}y = 0 \\ y(0) = \text{.....} \\ y'(0) = \text{.....} \end{cases} .$$

(p) *(6)* A given population of size p depends on the time $t \geq 0$ following the logistic law:

$$p' = 10p - 2p^2.$$

If $p(0) = 4$, then the solution of the previous differential equation is monotonic .... and $\lim_{t \to +\infty} p(t) = \text{.......}$

(q) *(4)* The graph of the solution of the IVP $\begin{cases} y'' = -4 \\ y'(0) = 0 \\ y(0) = 2 \end{cases}$ is

Part II

- Give your answers in exact form. For example, $\frac{\pi}{3}$ is an exact number while 1.047 is a decimal approximation for the same number.
- In order to receive credit, you must show all of your work. If you do not indicate the way in which you solve a problem, you may get little or no credit for it, even if your answer is correct.

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1. For $\alpha \in \mathbb{R}$, consider the following matrix $\mathbf{A} = \begin{bmatrix} \alpha & 0 & 4 \\ 0 & 1 & 0 \\ 4 & 0 & \alpha \end{bmatrix}$.

   (a) Classify the quadratic form $Q(X) = X^T\mathbf{A}X$, $X \in \mathbb{R}^3$, as function of α.

   (b) Find the value of α for which $(1, 0, 1)$ is an eigenvector of $\mathbf{A}$ associated to 4.

2. Consider the map $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = \begin{cases} \dfrac{x^2 + y^2}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.

   (a) Show that f is discontinuous at $(0, 0)$.

   (b) Is f differentiable at $(0, 0)$? Why?

   (c) **Using the Euler identity**, compute

   $$2\frac{\partial f}{\partial x}(2, 4) + 4\frac{\partial f}{\partial y}(2, 4).$$

3. Consider the polynomial map $f : \mathbb{R}^2 \to \mathbb{R}$ defined by:

   $$f(x, y) = (y - x^2)(y - 2)$$

   (a) Identify and classify the critical points of f.

   (b) Compute $f(x, 0)$ and conclude that f is unbounded.

4. Consider the set $\Omega = \{(x, y) \in \mathbb{R}^2 : y \leq x \wedge y \geq -x \wedge x \leq 2\}$. Compute

   $$\iint_\Omega x^2 e^{yx - x^2} \text{ dx dy}.$$

5. Consider the following differential equation (y is a function of x):

$$y'y = xy^2 - 4x.$$

(a) Show that the constant maps $y_1(x) = 2$ and $y_2(x) = -2$ are particular solutions of the differential equation.

(b) Write explicitly the solution of the IVP:

$$\begin{cases} y'y = xy^2 - 4x \\ y(1) = \sqrt{3} \end{cases}$$

and identify its maximal domain.

DO NOT DO THIS! :)

Credits:

I	II.1(a)	II.1(b)	II.2(a)	II.2(b)	II.2(c)	II.3(a)	II.3(b)	II.4	II.5(a)	II.5(b)
90	15	10	10	5	10	15	10	15	5	15