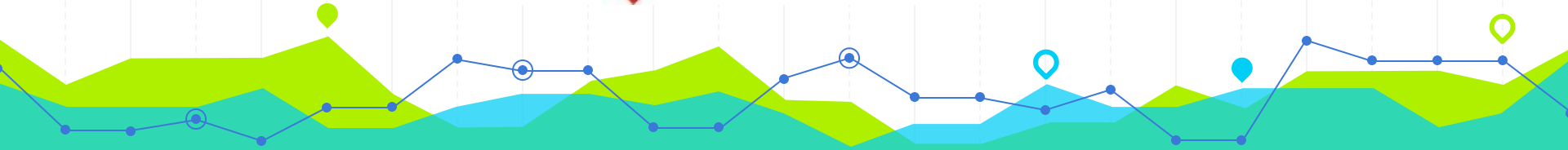




Lisbon School
of Economics
& Management
Universidade de Lisboa



STATISTICS I

Economics / Finance/ Management
2nd Year/2nd Semester
2024/2025

LESSON 5

Professor: Elisabete Fernandes

E-mail: efernandes@iseg.ulisboa.pt



<https://doity.com.br/estatistica-aplicada-a-nutricao>

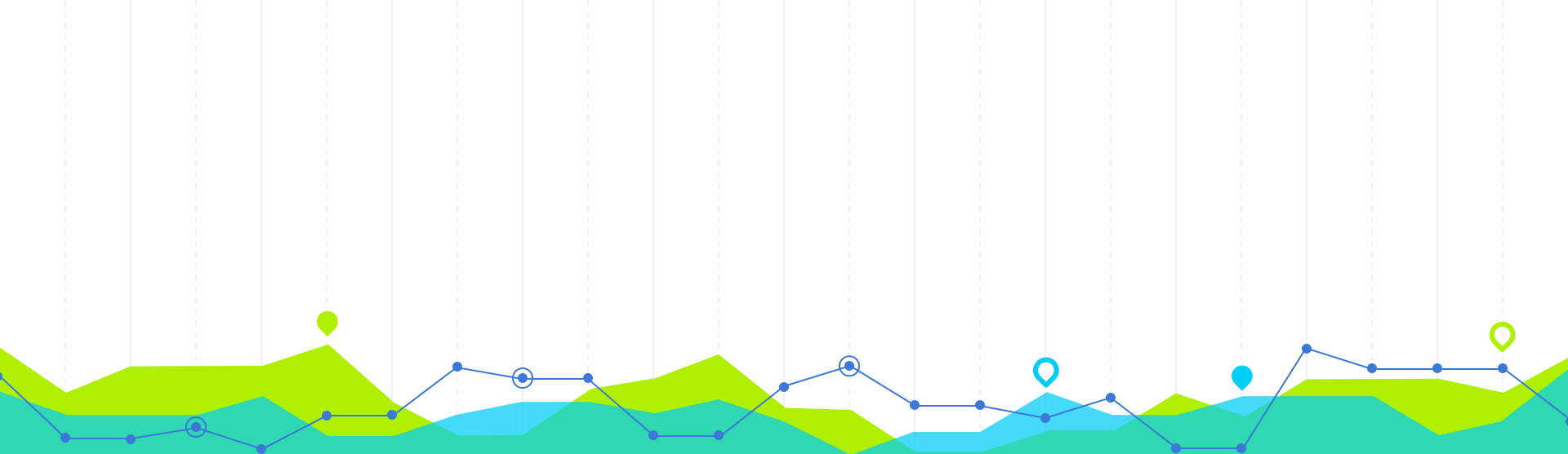


<https://basiccode.com.br/produto/informatica-basica/>

Roadmap:

- Probability
- Random variable and two dimensional random variables:
 - Distribution
 - Joint distribution
 - Marginal distribution
 - Conditional distribution functions
- Expectations and parameters for a random variable and two dimensional random variables
- Discrete Distributions
- Continuous Distributions

Bibliography: Miller & Miller, John E. (2014) Freund's Mathematical Statistics with applications, 8th Edition, Pearson Education, [MM]



Expected Value, Variance and Quantiles for a Random Variable: Exercises

Chapter 3

1

1. Let X be a random variable that takes on the values 0, 1, 2, and 3 with probabilities $\frac{1}{10}$, $\frac{3}{10}$, $\frac{2}{10}$, $\frac{4}{10}$.
- (a) Find $E(X)$ and $E(X^2)$.
 - (b) Use the results of part (a) to determine the value of $E[(X - \mu_X)^2]$.
 - (c) Use the definition to calculate σ_X .



Exercise 1 a)

$$D_x = \{0, 1, 2, 3\}$$

$$f_x(x) = \begin{cases} \frac{1}{10} & (x = 0) \\ \frac{3}{10} & (x = 1) \\ \frac{2}{10} & (x = 2) \\ \frac{4}{10} & (x = 3) \end{cases}$$

Exercise 1 a)

a)

$$E(x) = \sum_{x=0}^3 x f_x(x) = 0 \times \frac{1}{10} + 1 \times \frac{3}{10} + 2 \times \frac{2}{10} + 3 \times \frac{4}{10} =$$

$$= \frac{3}{10} + \frac{4}{10} + \frac{12}{10} = \frac{19}{10}$$

$$E(x^2) = \sum_{x=0}^3 x^2 f_x(x) = 0^2 \times \frac{1}{10} + 1^2 \times \frac{3}{10} + 2^2 \times \frac{2}{10} + 3^2 \times \frac{4}{10} =$$

$$= \frac{3}{10} + \frac{8}{10} + \frac{36}{10} = \frac{47}{10}$$

Exercise 1 b) and c)

b)

$$\begin{aligned} E[(X - \mu_X)^2] &= \text{Var}(X) = E(X^2) - E(X)^2 = \\ &= \frac{47}{10} - \left(\frac{19}{10}\right)^2 = \frac{470}{100} - \frac{361}{100} = \frac{109}{100} \end{aligned}$$

Note: A pink arrow points from the handwritten σ_X^2 to the $\text{Var}(X)$ term in the equation.

c)

$$\sigma_X = \sqrt{\sigma_X^2} = \sqrt{\frac{109}{100}} =$$

2. Let X be a continuous random variable and f_X its density function

$$f_X(x) = \begin{cases} 1/3, & 0 < x < 1 \\ \frac{4}{45}x, & 1 < x < 4 \end{cases}.$$

- (a) Compute the expected value and the variance of X .
- (b) Compute the expected value of Y that is given by

$$Y = g(X) = \begin{cases} 0, & X < 1 \\ 1, & X \geq 1 \end{cases}.$$

- (c) Compute the expected value of $Z = 2Y - 1$.



Exercise 2 a)

$$\begin{aligned} E(X) &= \int_0^4 x f_X(x) dx = \int_0^1 \frac{x}{3} dx + \int_1^4 \frac{4}{45} x^2 dx = \\ &= \frac{1}{3} \left[\frac{x^2}{2} \right]_0^1 + \frac{4}{45} \left[\frac{x^3}{3} \right]_1^4 = \frac{1}{3} \left(\frac{1}{2} - 0 \right) + \frac{4}{45} \left(\frac{4^3}{3} - \frac{1^3}{3} \right) = \\ &= \frac{1}{6} + \frac{4}{45} \left(\frac{64}{3} - \frac{1}{3} \right) = \frac{1}{6} + \frac{4}{45} \left(\frac{63}{3} \right) = \frac{1}{6} + \frac{252}{135} \\ &= \frac{135}{810} + \frac{1512}{810} = \frac{1647}{810} = \frac{61}{30} \end{aligned}$$

$$\text{Var}(X) = E(X^2) - E(X)^2 = \frac{52}{9} - \left(\frac{61}{30} \right)^2 = \frac{493}{300}$$

Auxiliary calculation:

$$\begin{aligned} E(X^2) &= \int_0^4 x^2 f_X(x) dx = \int_0^1 x^2 \cdot \frac{1}{3} dx + \int_1^4 x^2 \cdot \frac{4}{45} x dx = \frac{1}{3} \int_0^1 x^2 dx + \frac{4}{45} \int_1^4 x^3 dx = \\ &= \frac{1}{9} \left[x^3 \right]_0^1 + \frac{4}{180} \left[x^4 \right]_1^4 = \frac{1}{9} (1^3 - 0^3) + \frac{1}{45} (4^4 - 1^4) = \frac{1}{9} + \frac{255}{45} = \\ &= \frac{1}{9} + \frac{51}{9} = \frac{52}{9} \end{aligned}$$

Exercise 2 b)

b)

$$E[g(x)] = \int_{-\infty}^{+\infty} g(x) f_x(x) dx$$

$$Y = g(x) = \begin{cases} 0, & x < 1 \\ 1, & x \geq 1 \end{cases}$$

$$\begin{aligned} E(Y) &= \int_0^4 g(x) f_x(x) dx = \int_0^1 0 f_x(x) dx + \int_1^4 1 f_x(x) dx = \\ &= \underbrace{\int_0^1 0 dx}_0 + \int_1^4 \frac{4}{45} x dx = \frac{4}{45} \left[\frac{x^2}{2} \right]_1^4 = \\ &= \frac{4}{90} (4^2 - 1^2) = \frac{2}{45} \times 15 = \frac{30}{45} = \frac{6}{9} = \frac{2}{3} \end{aligned}$$

Exercise 2 c)

$$E(Z) = E(2Y - 1) = 2E(Y) - 1 = 2 \times \frac{2}{3} - 1 = \frac{4}{3} - 1 = \frac{1}{3}$$

Alternative solution:

$$\text{Let: } Y = g_1(x) = \begin{cases} 0, & x < 1 \\ 1, & x \geq 1 \end{cases} \quad \text{and}$$

$$Z = g_2(Y) = 2Y - 1$$

$$Y = \begin{cases} 0 \rightarrow Z(0) = -1 \\ 1 \rightarrow Z(1) = 1 \end{cases}$$

$$\text{Then } Z = \underbrace{g_2 \circ g_1}_{g}(x) = g_2(g_1(x)) = g(x) \begin{cases} -1 & \text{if } 0 < x < 1 \\ 1 & \text{if } 1 \leq x < 4 \end{cases}$$

$$E(Z) = E(g(x)) = \int_0^4 g(x) f_x(x) dx = \int_0^1 (-1) f_x(x) dx + \int_1^4 1 f_x(x) dx =$$

$$= \int_0^1 -\frac{1}{3} dx + \int_1^4 \frac{4}{45} x dx = -\frac{1}{3} [x]_0^1 + \frac{4}{45} \left[\frac{x^2}{2} \right]_1^4 =$$

$$= -\frac{1}{3} (1 - 0) + \frac{4}{90} (4^2 - 1^2) = -\frac{1}{3} + \frac{2}{45} (15) = -\frac{1}{3} + \frac{30}{45} =$$

$$= -\frac{5}{15} + \frac{10}{15} = \frac{5}{15} = \frac{1}{3}$$

3. The demand of a certain product, in Kg, in a random day is well represented by the random variable X with density function

$$f_X(x) = \begin{cases} 1/5, & 0 < x < 5 \\ 0, & \text{otherwise} \end{cases}.$$

The firm that sells this product has a profit of 5 euros per Kg sold and a loss of 2 euros per Kg that is not sold.

- (a) How many Kg of the product should the firm have in stock to maximize the expected profit?
- (b) Assume now that X is a discrete random variable, with a probability function

$$f_X(x) = \frac{1}{6}, \quad \text{for } x = 0, 1, 2, 3, 4, 5.$$

What is the expected profit?



Exercise 3 a)

$X \equiv$ Demand (in Kg)

$$f_x(x) = \begin{cases} \frac{1}{5} & (0 < x < 5) \\ 0 & (\text{otherwise}) \end{cases}$$

$Y =$ Profit (in €)

$Q =$ Quantity in stock ($Q \in \mathbb{R}$)

$$\begin{aligned} Y &= 5 \min(Q, X) - 2 \max(0, Q - X) = \\ &= \begin{cases} 5X - 2(Q - X) & (0 \leq X < Q) \\ 5Q & (Q \leq X \leq 5) \end{cases} = \begin{cases} 7X - 2Q & (0 \leq X < Q) \\ 5Q & (Q \leq X \leq 5) \end{cases} = g(X; Q) \end{aligned}$$

Exercise 3 a)

First let's find an expression for the expected profit:

$$\begin{aligned} E(Y) &= \int_0^5 g(x) f_x(x) dx = \int_0^Q (7x - 2Q) \frac{1}{5} dx + \int_Q^5 (5Q) \frac{1}{5} dx \\ &= \frac{1}{5} \int_0^Q 7x - 2Q dx + \int_Q^5 Q dx = \\ &= \frac{7}{10} [x^2]_0^Q - \frac{2Q}{5} [x]_0^Q + Q[x]_Q^5 = \\ &= \frac{7}{10} Q^2 - \frac{2Q}{5} Q^2 + (5-Q)Q = \frac{3}{10} Q^2 - Q^2 + 5Q \\ &= -\frac{7}{10} Q^2 + 5Q \end{aligned}$$

Exercise 3 a)

Now let's find the value of Q that maximizes $E(Y)$.

First order condition:

$$\frac{d}{dQ} E(Y) = \frac{d}{dQ} \left(-\frac{7}{10} Q^2 + 5Q \right) = -\frac{7}{5} Q + 5 = 0 \quad (\Rightarrow)$$

$$(\Rightarrow) \frac{7}{5} Q = 5 \quad (\Rightarrow) Q = \frac{25}{7} \rightarrow \text{critical point of } E(Y)$$

Second order condition:

$$\frac{d^2}{dQ^2} E(Y) = \frac{d}{dQ} \left(-\frac{7}{5} Q + 5 \right) = -\frac{7}{5} < 0$$

$Q = \frac{25}{7}$ is a maximizer of $E(Y)$. Since it is the only critical point, it is the global maximizer.

Conclusion: $Q = \frac{25}{7}$

Exercise 3 b)

$$f_x(x) = \frac{1}{6} \quad (0, 1, 2, 3, 4, 5)$$

$$Y = \begin{cases} 7X - 2Q & (0 \leq X < Q) \\ 5Q & (Q \leq X \leq 5) \end{cases} = g(x; Q)$$

$$E(Y) = \sum_{x=0}^5 g(x; Q) f_x(x) = \sum_{x=0}^{Q-1} (7x - 2Q) \frac{1}{6} + \sum_{x=Q}^5 5Q \frac{1}{6}$$

$$= \frac{7}{6} \sum_{x=0}^{Q-1} x - \frac{1}{3} \sum_{x=0}^{Q-1} Q + \frac{5}{6} \sum_{x=Q}^5 Q =$$

$$= \frac{7}{6} \left(\frac{1}{2} (Q-1)Q \right) - \frac{1}{3} Q^2 + \frac{5}{6} (6 - Q)Q =$$

$$= \frac{7}{12} (Q^2 - Q) - \frac{1}{3} Q^2 + \frac{5}{6} (6Q - Q^2) =$$

$$= \frac{7}{12} Q^2 - \frac{7}{12} Q - \frac{1}{3} Q^2 + \frac{30}{6} Q - \frac{5}{6} Q^2 =$$

$$= -\frac{7}{12} Q^2 + \frac{53}{12} Q \rightarrow \text{function of } Q$$

Exercise 3 b)

Conclusion:

Q	0	1	2	3	4	5
$E(Y)$	0	3.8(3)	6.5	8	8.3(3)	7.5

Auxiliary calculations:

$$\sum_{x=0}^{Q-1} x = \sum_{x=1}^{Q-1} x = \frac{1}{2} (Q-1) Q$$

arithmetic series

$$\sum_{x=0}^{Q-1} Q = \underbrace{Q + Q + Q + \dots + Q}_{Q \text{ times}} = Q^2$$

$$\sum_{x=Q}^5 Q = \underbrace{Q + Q + \dots + Q}_{5-Q+1 \text{ times} = 6-Q \text{ times}} = (6-Q) Q$$

4. Let X be a continuous random variable and f_X its distribution. Prove that the expected value of

$$Y = \begin{cases} a, & X < 0 \\ b, & X \geq 0 \end{cases},$$

is $E(Y) = aP(X < 0) + bP(X \geq 0)$.



Exercise 4

X is a continuous r.v.; $X \sim f_x(x)$.

$$Y = \begin{cases} a, & X < 0 \\ b, & X \geq 0 \end{cases} = g(x)$$

$$E(Y) = E[g(X)] = \int_{-\infty}^{+\infty} g(x) f_x(x) dx =$$

$$= \int_{-\infty}^0 a f_x(x) dx + \int_0^{+\infty} b f_x(x) dx =$$

$$= a P(X < 0) + b P(X \geq 0), \quad Q \in \mathbb{D}$$

$= P(X > 0)$ because X is
a continuous r.v.

5. Find $E(X)$, $E(X^2)$ and σ_X^2 for the random variable X that has probability density function

$$f_X(x) = \begin{cases} \frac{x}{2} & \text{for } 0 < x < 2 \\ 0 & \text{elsewhere} \end{cases}.$$



Exercise 5

$$f_x(x) = \begin{cases} \frac{x}{2} & (0 < x < 2) \\ 0 & (\text{elsewhere}) \end{cases}$$

$$\begin{aligned} E(X) &= \int_{-\infty}^{+\infty} x f_x(x) dx = \int_0^2 x \frac{x}{2} dx = \frac{1}{2} \left[\frac{x^3}{3} \right]_0^2 = \\ &= \frac{1}{6} (2^3 - 0^3) = \frac{8}{6} = \frac{4}{3} \end{aligned}$$

$$\begin{aligned} E(X^2) &= \int_{-\infty}^{+\infty} x^2 f_x(x) dx = \int_0^2 x^2 \frac{x}{2} dx = \frac{1}{2} \left[\frac{x^4}{4} \right]_0^2 = \\ &= \frac{1}{8} (16 - 0) = 2 \end{aligned}$$

$$\sigma_x^2 = E(X^2) - E(X)^2 = 2 - \left(\frac{4}{3}\right)^2 = \frac{18}{9} - \frac{16}{9} = \frac{2}{9}$$

6. Let X be a discrete random variable such that

$$f_X(x) = \begin{cases} 1/2, & x = 0 \\ 1/3, & x = 1 \\ 1/6, & x = 2 \end{cases}$$

Compute γ_1 .



Exercise 6

$$f_x(x) = \begin{cases} \frac{1}{2} & (x=0) \\ \frac{1}{3} & (x=1) \\ \frac{1}{6} & (x=2) \end{cases}$$

$$\gamma_1 = \frac{\mu_3}{\sigma_x^3} = \frac{E[(X - \mu_x)^3]}{\sigma_x^3}$$

$$\begin{aligned} \mu_x = E(X) &= \sum_{x=0}^2 x f_x(x) = \frac{1}{2} \times 0 + \frac{1}{3} \times 1 + \frac{1}{6} \times 2 = \\ &= \frac{1}{3} + \frac{2}{6} = \frac{2}{3} \end{aligned}$$

Exercise 6

$$\begin{aligned} E[\underbrace{(X - \mu_X)^3}_{g(X)}] &= \sum_{x=0}^2 (x - \mu_X)^3 f_X(x) = \left(0 - \frac{2}{3}\right)^3 \frac{1}{2} + \left(1 - \frac{2}{3}\right)^3 \frac{1}{3} + \left(2 - \frac{2}{3}\right)^3 \frac{1}{6} = \\ &= \left(-\frac{2}{3}\right)^3 \frac{1}{2} + \left(\frac{1}{3}\right)^3 \frac{1}{3} + \left(\frac{4}{3}\right)^3 \frac{1}{6} = \\ &= -\frac{8}{27} \times \frac{1}{2} + \frac{1}{27} \times \frac{1}{3} + \frac{64}{27} \times \frac{1}{6} = \\ &= -\frac{8}{54} + \frac{1}{81} + \frac{64}{162} = \\ &= -\frac{24}{162} + \frac{2}{162} + \frac{64}{162} = \frac{42}{162} = \frac{7}{27} \end{aligned}$$

Exercise 6

$$\sigma_x^2 = E(X^2) - E(X)^2 = 1 - \left(\frac{2}{3}\right)^2 = 1 - \frac{4}{9} = \frac{5}{9}$$

$$\begin{aligned} E(X^2) &= \sum_{x=0}^2 x^2 f_X(x) = \frac{1}{2} \times 0^2 + \frac{1}{3} \times 1^2 + \frac{1}{6} \times 2^2 = \\ &= \frac{1}{3} + \frac{4}{6} = 1 \end{aligned}$$

Conclusion:

$$\gamma_1 = \frac{\frac{7}{27}}{\left(\sqrt{\frac{5}{9}}\right)^3} \simeq 0.626099$$

7. Let X be a continuous random variable such that

$$f_X(x) = \begin{cases} x, & 0 < x < 1 \\ 1/2, & 1 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

Compute γ_2 .



Exercise 7

$$f_x(x) = \begin{cases} x, & 0 < x < 1 \\ 1/2, & 1 < x < 2 \\ 0, & (\text{otherwise}) \end{cases}$$

$$\gamma_2 = \frac{E[(X - \mu_X)^4]}{\text{Var}(X)^2} = \frac{\mu_4}{\sigma_X^4} = \frac{\frac{24594}{207360}}{\left(\frac{35}{144}\right)^2} = \frac{12297}{6125} \approx 2.00767$$

$$\begin{aligned} \mu_X &= \int_{-\infty}^{+\infty} x f_x(x) dx = \int_0^1 x^2 dx + \int_1^2 \frac{x}{2} dx = \\ &= \left[\frac{x^3}{3} \right]_0^1 + \frac{1}{2} \left[\frac{x^2}{2} \right]_1^2 = \frac{1}{3} + \frac{1}{4} (4-1) = \\ &= \frac{1}{3} + \frac{3}{4} = \frac{4}{12} + \frac{9}{12} = \frac{13}{12} \end{aligned}$$

$$\text{Var}(X) = E(X^2) - E(X)^2 = \frac{17}{12} - \left(\frac{13}{12}\right)^2 = \frac{17}{12} - \frac{169}{144} = \frac{204-169}{144} = \frac{35}{144}$$

$$\begin{aligned} E(X^2) &= \int_{-\infty}^{+\infty} x^2 f_x(x) dx = \int_0^1 x^3 dx + \int_1^2 \frac{x^2}{2} dx = \\ &= \left[\frac{x^4}{4} \right]_0^1 + \frac{1}{2} \left[\frac{x^3}{3} \right]_1^2 = \frac{1}{4} (1-0) + \frac{1}{6} (8-1) = \\ &= \frac{1}{4} + \frac{7}{6} = \frac{6}{24} + \frac{28}{24} = \frac{34}{24} = \frac{17}{12} \end{aligned}$$

Exercise 7

$$\begin{aligned}
 E[(X - \mu_X)^4] &= \int_{-\infty}^{+\infty} (x - \mu_X)^4 f_X(x) dx = \underbrace{\int_0^1 (x - \frac{13}{12})^4 x dx}_{(*)} + \underbrace{\int_1^2 (x - \frac{13}{12})^4 \frac{1}{2} dx}_{(*)} = \\
 &= \frac{11\,173}{207\,360} + \frac{13\,421}{207\,360} = \frac{24\,594}{207\,360}
 \end{aligned}$$

see auxiliary calculation

$$\begin{aligned}
 (*) \int_0^1 (x - \frac{13}{12})^4 x dx &= \int_0^1 (x^4 - \frac{13}{3} x^3 + \frac{169}{24} x^2 - \frac{2197}{432} x + \frac{28561}{20736}) x dx \\
 &= \int_0^1 x^5 - \frac{13}{3} x^4 + \frac{169}{24} x^3 - \frac{2197}{432} x^2 + \frac{28561}{20736} x dx \\
 &= \left[\frac{x^6}{6} \right]_0^1 - \frac{13}{3} \left[\frac{x^5}{5} \right]_0^1 + \frac{169}{24} \left[\frac{x^4}{4} \right]_0^1 - \frac{2197}{432} \left[\frac{x^3}{3} \right]_0^1 + \frac{28561}{20736} \left[\frac{x^2}{2} \right]_0^1 \\
 &= \frac{1}{6} - \frac{13}{3} \left(\frac{1}{5} \right) + \frac{169}{24} \left(\frac{1}{4} \right) - \frac{2197}{432} \left(\frac{1}{3} \right) + \frac{28561}{20736} \left(\frac{1}{2} \right) \\
 &= \frac{11\,173}{207\,360}
 \end{aligned}$$

Exercise 7

$$\begin{aligned}
 (*) \int_1^2 \left(x - \frac{13}{12}\right)^4 \frac{1}{2} dx &= \int_1^2 \left(x^4 - \frac{13}{3}x^3 + \frac{169}{24}x^2 - \frac{2197}{432}x + \frac{28561}{20736}\right) \frac{1}{2} dx \\
 &= \frac{1}{2} \int_1^2 x^4 - \frac{13}{3}x^3 + \frac{169}{24}x^2 - \frac{2197}{432}x + \frac{28561}{20736} dx \\
 &= \frac{1}{2} \left(\left[\frac{x^5}{5}\right]_1^2 - \frac{13}{3} \left[\frac{x^4}{4}\right]_1^2 + \frac{169}{24} \left[\frac{x^3}{3}\right]_1^2 - \frac{2197}{432} \left[\frac{x^2}{2}\right]_1^2 + \frac{28561}{20736} [x]_1^2 \right) \\
 &= \frac{1}{10} (2^5 - 1) - \frac{13}{24} (2^4 - 1) + \frac{169}{144} (2^3 - 1) - \frac{2197}{1728} (2^2 - 1) + \frac{28561}{20736} (2 - 1) = \\
 &= \frac{13421}{207360}
 \end{aligned}$$

Auxiliary calculation:

$$\begin{aligned}
 \left(x - \frac{13}{12}\right)^4 &\stackrel{\text{Binomial theorem (Newton)}}{=} \sum_{k=0}^4 \binom{4}{k} x^{4-k} \left(-\frac{13}{12}\right)^k = \binom{4}{0} x^4 \underbrace{\left(-\frac{13}{12}\right)^0}_1 + \binom{4}{1} x^3 \underbrace{\left(-\frac{13}{12}\right)^1}_{-1} + \binom{4}{2} x^2 \underbrace{\left(-\frac{13}{12}\right)^2}_6 + \binom{4}{3} x \underbrace{\left(-\frac{13}{12}\right)^3}_{-4} + \binom{4}{4} x^0 \underbrace{\left(-\frac{13}{12}\right)^4}_{1} = \\
 &= x^4 - 4x^3 \frac{13}{12} + 6x^2 \left(\frac{13}{12}\right)^2 - 4x \left(\frac{13}{12}\right)^3 + \left(\frac{13}{12}\right)^4 = \\
 &= x^4 - \frac{52}{12} x^3 + \frac{1014}{144} x^2 - \frac{8788}{1728} x + \frac{28561}{20736} = \\
 &= x^4 - \frac{13}{3} x^3 + \frac{169}{24} x^2 - \frac{2197}{432} x + \frac{28561}{20736}
 \end{aligned}$$

Exercise 7

Note: In the last integral, instead of the Binomial theorem we could have used integration by substitution, which would have been more convenient:

$$\int_1^2 \left(x - \frac{13}{12}\right)^4 \frac{1}{2} dx = \frac{1}{2} \int_1^2 \underbrace{\left(x - \frac{13}{12}\right)}_u \underbrace{dx}_{du=1} = \frac{1}{2} \int_{-\frac{1}{12}}^{\frac{11}{12}} u^4 du = \frac{1}{10} [u^5]_{-\frac{1}{12}}^{\frac{11}{12}} = \frac{1}{10} \left(\left(\frac{11}{12}\right)^5 - \left(-\frac{1}{12}\right)^5 \right) = \frac{13421}{207360}$$

$$u = x - \frac{13}{12} \begin{cases} \rightarrow x = 2 \rightarrow u = \frac{11}{12} \\ \rightarrow x = 1 \rightarrow u = -\frac{1}{12} \end{cases}$$

8. Let X be a random variable that has probability density function

$$f(x) = \begin{cases} x/2 & \text{for } 0 < x \leq 1 \\ 1/2 & \text{for } 1 < x \leq 2 \\ (3-x)/2 & \text{for } 2 < x < 3 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Find $E(X)$, the median and the mode of X .

(b) Find $E(X^2)$.

(c) Use the results of part (a) and (b) to determine $E(X^2 - 5X + 3)$.

(d) Compute the standard deviation.

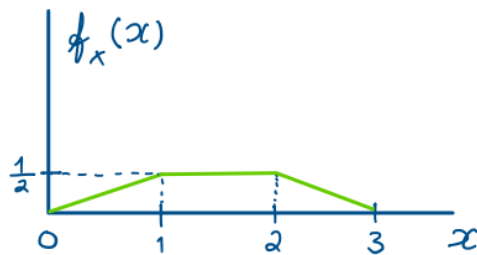


Exercise 8 a)

$$f(x) = \begin{cases} x/2 & \text{for } 0 < x \leq 1 \\ 1/2 & \text{for } 1 < x \leq 2 \\ (3-x)/2 & \text{for } 2 < x < 3 \\ 0 & \text{elsewhere} \end{cases}$$

$$\begin{aligned} a) \quad \epsilon(x) &= \int_{-\infty}^{+\infty} x f_x(x) dx = \int_0^1 x \frac{x}{2} dx + \int_1^2 x \frac{1}{2} dx + \int_2^3 x \frac{3-x}{2} dx = \\ &= \frac{1}{2} \int_0^1 x^2 dx + \frac{1}{2} \int_1^2 x dx + \frac{1}{2} \int_2^3 3x - x^2 dx = \\ &= \frac{1}{2} \left[\frac{x^3}{3} \right]_0^1 + \frac{1}{2} \left[\frac{x^2}{2} \right]_1^2 + \frac{1}{2} \left[\frac{3}{2} x^2 - \frac{x^3}{3} \right]_2^3 = \\ &= \frac{1}{6} + \frac{3}{4} + \frac{1}{2} \left(\frac{3}{2} 3^2 - \frac{3^3}{3} - \left(\frac{3}{2} 2^2 - \frac{2^3}{3} \right) \right) = \\ &= \frac{1}{6} + \frac{3}{4} + \frac{1}{2} \left(\frac{27}{2} - 9 - 6 + \frac{8}{3} \right) = \\ &= \frac{1}{6} + \frac{3}{4} + \frac{27}{4} - \frac{15}{2} + \frac{8}{6} = \frac{4}{24} + \frac{18}{24} + \frac{162}{24} - \frac{180}{24} + \frac{32}{24} = \\ &= \frac{36}{24} = \frac{6}{4} = \frac{3}{2} \end{aligned}$$

Exercise 8 a)



The median of X is a value $m \in \mathbb{R}$: $F_X(m) = \frac{1}{2}$

$$\int_0^1 f_X(x) dx = \int_0^1 \frac{x}{2} dx = \frac{1}{2} \left[\frac{x^2}{2} \right]_0^1 = \frac{1}{4} < \frac{1}{2}$$

$$\int_0^2 f_X(x) dx = \frac{1}{4} + \int_1^2 \frac{1}{2} dx = \frac{1}{4} + \frac{1}{2} [x]_1^2 =$$

$$= \frac{1}{4} + \frac{1}{2} = \frac{3}{4} > \frac{1}{2} \rightarrow 1 < m < 2$$

Exercise 8 a)

$$F_X(m) = \frac{1}{2} \quad (\Leftrightarrow) \quad \int_0^m f_X(x) dx = \frac{1}{2} \quad (\Leftrightarrow)$$

$$(\Leftrightarrow) \quad \frac{1}{4} + \int_1^m \frac{1}{2} dx = \frac{1}{2} \quad (\Leftrightarrow) \quad \frac{2}{4}$$

$$(\Leftrightarrow) \quad \frac{1}{2} [x]_1^m = \frac{1}{2} - \frac{1}{4} \quad (\Leftrightarrow)$$

$$(\Leftrightarrow) \quad \frac{1}{2} (m-1) = \frac{1}{4} \quad (\Leftrightarrow)$$

$$(\Leftrightarrow) \quad m-1 = \frac{1}{2} \quad (\Leftrightarrow) \quad m = \frac{3}{2}$$

- The mode of X , $\text{mo}(X)$, is a value such that $f_X(\text{mo}(X)) \geq f_X(x) \quad \forall x \in \mathbb{R}$
- $\text{mo}(X)$ does not have to be unique.

By looking at the plot we can easily see that
in this case we have $\text{mo}(X) = x : x \in (1, 2)$.

Another expression for the modes of X is: $\text{mo}(X) = \alpha + 2(1-\alpha)$, $\alpha \in (0, 1)$

Exercise 8 b)

$$\begin{aligned}\epsilon(X^2) &= \int_{-\infty}^{+\infty} x^2 f_X(x) dx = \int_0^1 x^2 \frac{x}{2} dx + \int_1^2 x^2 \frac{1}{2} dx + \int_2^3 x^2 \frac{3-x}{2} dx = \\&= \frac{1}{2} \int_0^1 x^3 dx + \frac{1}{2} \int_1^2 x^2 dx + \frac{1}{2} \int_2^3 (3x^2 - x^3) dx = \\&= \frac{1}{2} \left[\frac{x^4}{4} \right]_0^1 + \frac{1}{2} \left[\frac{x^3}{3} \right]_1^2 + \frac{1}{2} \left[x^3 - \frac{x^4}{4} \right]_2^3 = \\&= \frac{1}{8} + \frac{7}{6} + \frac{1}{2} \left(27 - \frac{81}{4} - (8 - 4) \right) = \\&= \frac{1}{8} + \frac{7}{6} + \frac{1}{2} \left(27 - \frac{81}{4} - 4 \right) = \\&= \frac{1}{8} + \frac{7}{6} + \frac{27}{2} - \frac{81}{8} - 2 = \\&= -\frac{80}{8} + \frac{7}{6} + \frac{81}{6} - \frac{12}{6} = -\frac{80}{8} + \frac{76}{6} = \\&= -\frac{480}{48} + \frac{608}{48} = \frac{128}{48} = \frac{16}{6} = \frac{8}{3}\end{aligned}$$

$\nearrow$
 $\div 8$

Exercise 8 c)

$$\begin{aligned} E(X^2 - 5X + 3) &= E(X^2) - 5E(X) + 3 = \\ &= \frac{8}{3} - 5\left(\frac{3}{2}\right) + 3 = \\ &= \frac{8}{3} - \frac{15}{2} + 3 = \\ &= \frac{16}{6} - \frac{45}{6} + \frac{18}{6} = -\frac{11}{6} \end{aligned}$$

Exercise 8 d)

d)

$$\sigma_x^2 = E(X^2) - E(X)^2 = \frac{8}{3} - \left(\frac{3}{2}\right)^2 = \frac{8}{3} - \frac{9}{4} =$$

$$= \frac{32}{12} - \frac{27}{12} = \frac{5}{12}$$

$$\sigma_x = \sqrt{\sigma_x^2} = \sqrt{\frac{5}{12}}$$

9. Find the expected value, the median and the mode of the discrete random variable X having the probability distribution $f_X(x) = |x - 2|/7$, $x = -1, 0, 1, 3$.



Exercise 9

$$f_x(x) = \frac{|x-2|}{7} \quad x = -1, 0, 1, 3$$

$$f_x(-1) = \frac{|-3|}{7} = \frac{3}{7} \quad f_x(1) = \frac{1}{7}$$

$$f_x(0) = \frac{|-2|}{7} = \frac{2}{7} \quad f_x(3) = \frac{1}{7}$$

$$E(x) = \sum_{x=-1}^3 x f_x(x) dx =$$

$$= (-1) \frac{3}{7} + 0 \times \frac{2}{7} + 1 \times \frac{1}{7} + 3 \times \frac{1}{7} =$$

$$= -\frac{3}{7} + \frac{1}{7} + \frac{3}{7} = \frac{1}{7}$$

Exercise 9

$$\text{mo}(X) = \arg \max_{x \in \mathbb{R}} f_x(x) = -1$$

$$\text{me}(X) = \min \{x \in \mathbb{R} : F_x(x) \geq 0.5\} = 0$$

Auxiliary calculations:

$$F_x(-1) = f_x(-1) = \frac{3}{7} < 0.5$$

$$F_x(0) = f_x(-1) + f_x(0) = \frac{5}{7} > 0.5$$

Thanks!

Questions?

