Foundations of Financial Economics
Bernardino Adão
Regular Exam
May, 7, 2025
Total time: 2 hours. Total points: 20

Instructions:

Please sit in alternate seats. This is a closed-book, closed-note exam. Please get rid of everything but pen/pencil. In your answer explain all the steps in your reasoning. Keep answers short; I don't give more credit for long answers, and I can take points off if you add things that are wrong or irrelevant.

Formulas:

If x and y are random variables then

$$E(xy) = ExEy + cov(x, y); \; \sigma^2(x) = Ex^2 - (Ex)^2,$$

$$\sigma^2(ax + by) = a^2\sigma^2(x) + b^2\sigma^2(y) + 2abcov(y, x),$$

$$cov(ax, by) = abcov(x, y); \; \sigma^2(kx) = k^2\sigma^2(x)$$

$$-1 \leq corr(x, y) \leq 1; \; corr(x, y) = \frac{cov(x, y)}{\sigma(x)\sigma(y)}$$

If x is normal distributed then $\exp(x)$ is lognormal, and

$$E\exp(x) = \exp(Ex + 0.5\sigma^2(x))$$

I (12 pts.)

1. This group of questions concerns basic concepts.

(1 pt) a. What is a stochastic discount factor?

(1 pt) b. What is the role of the stochastic discount factor in pricing equities?

(1 pt) c. What is the equity premium puzzle?

(1 pt) d. Explain how Epstein and Zin solve the equity premium puzzle and the risk free rate puzzle.

(1 pt) e. What is a geometric Brownian motion? Give an example of its use in financial economics.

(1 pt) f. What is the mean variance frontier portfolio?

(1 pt) g. What is the CAPM?

(1 pt) h. How does the Fama-Fench 3 factor model improve upon the CAPM?

(1 pt) i. What is the put-call parity?

(1 pt) j. Give an example of how options can be used to insure against an unexpected drop in the stock market.

(1 pt) k. Explain with an example how we can complete the markets with options.

II (6 pts.)

Consider a model with power utility, $u(C_t) = \frac{C_t^{1-\gamma}}{1-\gamma}$, and $\beta = 1$. Assume that $\ln\left(\frac{C_{t+1}}{C_t}\right) \equiv \Delta c_{t+1}$ is normal distributed. Let m_{t+1} denote the stochastic discount factor.

(2 pts) a. Show that the ratio

$$\frac{\sigma\left(m_{t+1}\right)}{E(m_{t+1})}$$

is approximately equal to $\gamma\sigma\left(\Delta c_{t+1}\right)$.

Answer:

$$m_{t+1} = \left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}$$

$$\Longrightarrow m_{t+1} \text{ is lognormal distributed}$$

$$Em_{t+1} = e^{E(\log m_{t+1})+0.5\sigma^2(\log m_{t+1})}$$

$$\begin{aligned} \sigma^2\left(m_{t+1}\right) &= E\left(m_{t+1}\right)^2 - \left(Em_{t+1}\right)^2 \\ &= e^{2E(\log m_{t+1})+2\sigma^2(\log m_{t+1})} \\ &\quad -e^{2E(\log m_{t+1})+\sigma^2(\log m_{t+1})} \end{aligned}$$

Thus,

$$\begin{aligned} \frac{\sigma^2\left(m_{t+1}\right)}{\left(E(m_{t+1})\right)^2} &= \frac{e^{2E(\log m_{t+1})+2\sigma^2(\log m_{t+1})} - e^{2E(\log m_{t+1})+\sigma^2(\log m_{t+1})}}{e^{2E(\log m_{t+1})+\sigma^2(\log m_{t+1})}} \\ &= e^{\sigma^2(\log m_{t+1})} - 1 \end{aligned}$$

Now use the approximation (first order Taylor approximation)

$$e^x \approx 1 + x$$

Replace above to get

$$\frac{\sigma^2\left(m_{t+1}\right)}{\left(E(m_{t+1})\right)^2} \approx \sigma^2\left(\log m_{t+1}\right),$$

or

$$\frac{\sigma\left(m_{t+1}\right)}{E(m_{t+1})} \approx \gamma\sigma\left(\Delta c_{t+1}\right).$$

(2 pts) b. Consider the market Sharpe ratio

$$\left|\frac{ER_m - R_f}{\sigma(R_m)}\right| \le \frac{\sigma\left(m\right)}{E(m)}$$

where R_m is the return on the market portfolio. The postwar NYSE index excess return is around 8% per year, with standard deviation around 16%. The standard deviation of log consumption growth is about 1%. Is this a challenge to the model? Explain.

Answer:

$$0.5 = \frac{0.08}{0.16} = \frac{ER_m - R_f}{\sigma(R_m)} \leq \gamma\sigma\left(\Delta c_{t+1}\right) = 0.01\gamma$$

Need to assume that the relative risk aversion is extremely large, i.e. $\gamma > 50$.

(2 pts) c. In the data $E_t\left(\Delta c_{t+1}\right) = 2$. What do the data and the model imply for the riskless interest rate? Discuss.

Answer:

$$\left(R_{t+1}^f\right)^{-1} = E_t(m_{t+1})$$

$$\begin{aligned}\left(R_{t+1}^f\right)^{-1} &= E_t e^{\ln m_{t+1}} = E_t e^{\ln(C_{t+1}/C_t)^{-\gamma}} = E_t e^{-\gamma\Delta c_{t+1}} \\ &= e^{-\gamma E_t(\Delta c_{t+1}) + 0.5\gamma^2\sigma_t^2(\Delta c_{t+1})}\end{aligned}$$

or

$$-r_{t+1}^f \equiv -\ln R_{t+1}^f = -\gamma E_t\left(\Delta c_{t+1}\right) + 0.5\gamma^2\sigma_t^2\left(\Delta c_{t+1}\right)$$

As $\gamma > 50$ then $\gamma E_t\left(\Delta c_{t+1}\right) > 100$ while in the data is the risk free interest rate is small. Also, as γ is large then fluctuations in $E_t\left(\Delta c_{t+1}\right)$ or $\sigma_t^2\left(\Delta c_{t+1}\right)$ lead to large fluctuations in r_{t+1}^f which is something we do not observe in the data. In the data the risk free interest rate is pretty stable.

III (2 pts.)

Let $\mathcal{B}_{j,t}$ be the price at t of a bond that matures at $t+j$. These bonds pay no dividends and at maturity these bonds are worth $\mathcal{B}_{0,t} = 1$. The pricing equation for $\mathcal{B}_{j,t}$ is

$$\mathcal{B}_{j,t} = E_t\left(m_{t+k}\mathcal{B}_{j-k,t+k}\right),$$

where the stochastic discount factor is $m_{t+k} = \beta^k\left(\frac{C_{t+k}}{C_t}\right)^{-\gamma}$, C_t is the consumption of the representative consumer at date t, $0 < \beta < 1$ and $\gamma > 1$.

(1 pt) a. Let the one-period riskless return between date t and $t+1$ be denoted by $R_{1,t}^f$, and the two-period riskless return between date t and $t+2$ be denoted by $R_{2,t}^f$. Show that $R_{2,t}^f$ satisfies the equation:

$$\frac{1}{R_{2,t}^f} = \frac{1}{R_{1,t}^f}E_t\left[\frac{1}{R_{1,t+1}^f}\right] + \beta^2 cov_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}, \left(\frac{C_{t+2}}{C_{t+1}}\right)^{-\gamma}\right].$$

Answer:

$$\mathcal{B}_{2,t} = E_t\left(m_{t+2}\mathcal{B}_{0,t+2}\right)$$

$$\Rightarrow \quad \frac{1}{R_{2,t}^{f}} = E_t\left(m_{t+2}\right) = E_t\left(\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\beta\left(\frac{C_{t+2}}{C_{t+1}}\right)^{-\gamma}\right)$$

$$\Longleftrightarrow \quad \frac{1}{R_{2,t}^{f}} = E_t\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} E_t\beta\left(\frac{C_{t+2}}{C_{t+1}}\right)^{-\gamma} + \beta^2 cov_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}, \left(\frac{C_{t+2}}{C_{t+1}}\right)^{-\gamma}\right]$$

$$\Longleftrightarrow \quad \frac{1}{R_{2,t}^{f}} = \frac{1}{R_{1,t}^{f}} E_t\left[\frac{1}{R_{1,t+1}^{f}}\right] + \beta^2 cov_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}, \left(\frac{C_{t+2}}{C_{t+1}}\right)^{-\gamma}\right]$$

(1 pt) b. Let $\beta^2 cov_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}, \left(\frac{C_{t+2}}{C_{t+1}}\right)^{-\gamma}\right]$ be the inverse of the term (or maturity) premium. Assume that $y_{t+1} \equiv \left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}$ is given by

$$y_{t+1} = \rho y_t + (1-\rho)\,\varepsilon_{t+1}$$

where $0 < \rho < 1$, and $\varepsilon_{t+1} \sim i.i.d.\ N\left(\mu, \sigma^2\right)$. What is the expression for the inverse of the term premium?

Answer:

$$\begin{aligned}
cov_t\left(y_{t+1}, y_{t+2}\right) &= E_t\left(y_{t+1}y_{t+2}\right) - E_t y_{t+1} E_t y_{t+2} \\
&= \rho E_t\left(y_{t+1}\right)^2 + (1-\rho)\rho y_t \mu + (1-\rho)^2\mu^2 \\
&\quad -\rho E_t y_{t+1} E_t y_{t+1} - E_t y_{t+1}\left(1-\rho\right) E_t \varepsilon_{t+2} \\
&= 0
\end{aligned}$$