STATISTICAL METHODS

Master in Industrial Management,

Operations and Sustainability (MIMOS)

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https://doity.com.br/estatistica-aplicada-a-nutricao

https://basiccode.com.br/produto/informatica-basica/

PROGRAM

1. Fundamental Concepts of Statistics
2. Descriptive Data Analysis
3. Introduction to Inferential Analysis
4. Parametric Hypothesis Testing
5. Non-Parametric Hypothesis Testing
6. Linear Regression Analysis

LECTURE 9 HOMEWORK: QUESTIONS AND SOLUTIONS

EXERCISE 6.32

6.32 A record store owner finds that 20% of customers entering her store make a purchase. One morning 180 people, who can be regarded as a random sample of all customers, enter the store.

a. What is the mean of the distribution of the sample proportion of customers making a purchase?
b. What is the variance of the sample proportion?
c. What is the standard error of the sample proportion?
d. What is the probability that the sample proportion is less than 0.15?

Newbold et al (2013)

EXERCISE 6.32 A): SOLUTION

Answer:

Given: population proportion $p = 0.20$, sample size $n = 180$.

For the sample proportion $\hat{p}$:

$$\hat{p} \sim N\left(p, \frac{p(1-p)}{n}\right), \quad SE = \sqrt{\frac{p(1-p)}{n}}$$

(a) Mean of the sampling distribution

$$E(\hat{p}) = p = 0.20$$

EXERCISE 6.32 B): SOLUTION

✅ Answer:

Given: population proportion $p = 0.20$, sample size $n = 180$.

For the sample proportion $\hat{p}$:

$$\hat{p} \sim N\left(p, \frac{p(1-p)}{n}\right), \quad SE = \sqrt{\frac{p(1-p)}{n}}$$

(b) Variance of the sampling distribution

$$Var(\hat{p}) = \frac{p(1-p)}{n} = \frac{0.20 \cdot 0.80}{180} = \frac{0.16}{180} \approx 0.0008889$$

EXERCISE 6.32 C): SOLUTION

Answer:

Given: population proportion $p = 0.20$, sample size $n = 180$.

For the sample proportion $\hat{p}$:

$$\hat{p} \sim N\left(p, \frac{p(1-p)}{n}\right), \quad SE = \sqrt{\frac{p(1-p)}{n}}$$

(c) Standard error of the sample proportion

Standard Deviation of $\hat{p}$

$$SE = \sqrt{0.0008889} \approx 0.0298$$

EXERCISE 6.32 D): SOLUTION

Answer:

Given: population proportion $p = 0.20$, sample size $n = 180$.

For the sample proportion $\hat{p}$:

Standard Deviation of $\widehat{\boldsymbol{p}}$

$$\hat{p} \sim N\left(p, \frac{p(1-p)}{n}\right), \quad SE = \sqrt{\frac{p(1-p)}{n}}$$

(d) Probability that $\hat{p} < 0.15$

Compute z-score:

Standard Normal Distribution Table

$$z = \frac{0.15 - 0.20}{0.0298} = \frac{-0.05}{0.0298} \approx -1.6779$$

$$P(\hat{p} < 0.15) = \Phi(-1.6779) \approx 0.0467$$

Central Limit Theorem: If n ≥ 25 then $Z = \frac{\widehat{p} - P}{\sqrt{\frac{P(1-P)}{n}}}$ ~ Normal(0, 1)

EXERCISE 6.48

6.48 A random sample of size $n = 25$ is obtained from a normally distributed population with a population mean of $\mu = 198$ and a variance of $\sigma^2 = 100$.

a. What is the probability that the sample mean is greater than 200?

b. What is the value of the sample variance such that 5% of the sample variances would be less than this value?

c. What is the value of the sample variance such that 5% of the sample variances would be greater than this value?

Newbold et al (2013)

EXERCISE 6.48 A): SOLUTION

Answer:

Given: $\mu = 198,\ \sigma^2 = 100\ (\sigma = 10),\ n = 25.$

(a) $P(\bar{X} > 200)$

$\bar{X} \sim$ Normal($\mu, \sigma^2/n$)

$$\bar{X} \sim N\left(198, \frac{\sigma^2}{n}\right) = N\left(198, \frac{100}{25}\right) = N(198, 4),$$

Standard Deviation of $\bar{X}$

so $\mathrm{sd}(\bar{X}) = 2.$

$$P(\bar{X} > 200) = P\left(Z > \frac{200 - 198}{2}\right) = P(Z > 1) = 1 - \Phi(1) \approx 0.158655\ (\approx 0.158$$

Standard Normal Distribution Table

Answer (a): $P(\bar{X} > 200) \approx 0.1587.$

EXERCISE 6.48 B): SOLUTION

Answer:

Given: $\mu = 198,\ \sigma^2 = 100\ (\sigma = 10),\ n = 25.$

(b) Value s^2_{low} such that $P(S^2 < s^2_{\text{low}}) = 0.05$

Let $\chi^2_{0.05,24}$ denote the 5th percentile of χ^2_{24}. Numerically $\chi^2_{0.05,24} \approx 13.8484$. Then

Chi-Square Distribution Table

$$s^2_{\text{low}} = \sigma^2 \frac{\chi^2_{0.05,24}}{n-1} = 100 \cdot \frac{13.8484}{24} \approx 57.70.$$

(sample standard deviation $\sqrt{57.70} \approx 7.596$.)

Answer (b): $s^2_{\text{low}} \approx 57.70$ (so $S \approx 7.60$).

$\mathbf{Q} = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{(n-1)}$

Alternative Solution:

$P(S^2 < a) = 0.05 \Leftrightarrow P\left(\frac{(n-1)S^2}{\sigma^2} < \frac{(n-1)a}{\sigma^2}\right) = 0.05 \Leftrightarrow$

$P\left(Q < \frac{24\times a}{100}\right) = 0.05$

Then $\frac{24\times a}{100} = \chi^2_{0.05;24} = 13.8484$

$\Leftrightarrow$ a = 100 × 13.8484/ 24 = 57.70

EXERCISE 6.48 C): SOLUTION

Answer:

Given: $\mu = 198,\ \sigma^2 = 100\ (\sigma = 10),\ n = 25.$

(c) Value s^2_{high} such that $P(S^2 > s^2_{\text{high}}) = 0.05$

$\mathbf{Q} = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{(n-1)}$

Let $\chi^2_{0.95,24} \approx 36.4150$. Then

Chi-Square Distribution Table

$$s^2_{\text{high}} = 100 \cdot \frac{36.4150}{24} \approx 151.73.$$

(sample standard deviation $\sqrt{151.73} \approx 12.32$.)

Answer (c): $s^2_{\text{high}} \approx 151.73$ (so $S \approx 12.32$).

Alternative Solution:

$P(S^2 > a) = 0.05 \Leftrightarrow P\left(\frac{(n-1)S^2}{\sigma^2} > \frac{(n-1)a}{\sigma^2}\right) = 0.05 \Leftrightarrow$

$P\left(Q < \frac{24 \times a}{100}\right) = 0.95$

Then $\frac{24 \times a}{100} = \chi^2_{0.95;24} = 36.4150$

$\Leftrightarrow$ a = 100 × 36.4150/ 24 = 151.73

EXERCISE 7.13

7.13 A personnel manager has found that historically the scores on aptitude tests given to applicants for entry-level positions follow a normal distribution with a standard deviation of 32.4 points. A random sample of nine test scores from the current group of applicants had a mean score of 187.9 points.

a. Find an 80% confidence interval for the population mean score of the current group of applicants.
b. Based on these sample results, a statistician found for the population mean a confidence interval extending from 165.8 to 210.0 points. Find the confidence level of this interval.

Newbold et al (2013)

EXERCISE 7.13 A): SOLUTION

Answer:

Given: $\sigma = 32.4,\ n = 9,\ \bar{x} = 187.9.$

Because the population standard deviation σ is known and the population is assumed normal, we use the normal (z) distribution.

(a) 80% confidence interval for μ

$$IC_{(1-\alpha)}(\mu) = \left(\bar{x} - z_{1-\frac{\alpha}{2}} \times \frac{\sigma}{\sqrt{n}}, \bar{x} + z_{1-\frac{\alpha}{2}} \times \frac{\sigma}{\sqrt{n}}\right)$$

Standard error:

$$SE = \frac{\sigma}{\sqrt{n}} = \frac{32.4}{\sqrt{9}} = \frac{32.4}{3} = 10.8.$$

Critical value for an 80% two-sided interval: $z_{0.90} = 1.28155$.

1-α = 0.80 ⇔ α = 0.20

1- α/2 = 0.90

$z_{1-\frac{\alpha}{2}} = z_{0.90} = 1.28155$

Standard Normal Distribution Table

Margin of error:

$$ME = z_{0.90}\ SE = 1.28155 \times 10.8 \approx 13.8427.$$

Confidence interval:

$$\bar{x} \pm ME = 187.9 \pm 13.8427$$

$$\boxed{80\%\ \text{CI}:\ [174.06,\ 201.74]\ \text{(approximately)}}$$

EXERCISE 7.13 B): SOLUTION

✅ Answer:

Given: $\sigma = 32.4,\ n = 9,\ \bar{x} = 187.9.$

Because the population standard deviation σ is known and the population is assumed normal, we use the normal (z) distribution.

$$IC_{(1-\alpha)}(\mu) = \left(\bar{x} - z_{1-\frac{\alpha}{2}} \times \frac{\sigma}{\sqrt{n}}, \bar{x} + z_{1-\frac{\alpha}{2}} \times \frac{\sigma}{\sqrt{n}}\right)$$

(b) Confidence level for the interval $[165.8,\ 210.0]$

The interval is centered at $\bar{x} = 187.9$. Margin of error:

$$ME = 210.0 - 187.9 = 22.1.$$

Compute the corresponding z:

$$z_{1-\frac{\alpha}{2}} = \frac{ME}{SE} = \frac{22.1}{10.8} \approx 2.0463.$$

The central probability is $P(-z < Z < z) = 2\Phi(z) - 1$. Using $\Phi(2.0463) \approx 0.979636$,

$$\text{Confidence level} = 2(0.979636) - 1 \approx 0.95927 \approx 95.93\%.$$

Alternative Solution:

ME = $z_{1-\frac{\alpha}{2}} \times \frac{\sigma}{\sqrt{n}}$ = (210.0 – 165.8)/2 = 22.1

Then $z_{1-\frac{\alpha}{2}} \times \frac{\sigma}{\sqrt{n}}$ = 22.1 ⇔ $z_{1-\frac{\alpha}{2}}$= 22.1 × $\sqrt{n}$/σ=2.0463

Using Standard Normal Distribution Table:

$z_{0.979636}$= 2.0463, then $1 - \frac{\alpha}{2} = 0.979636$

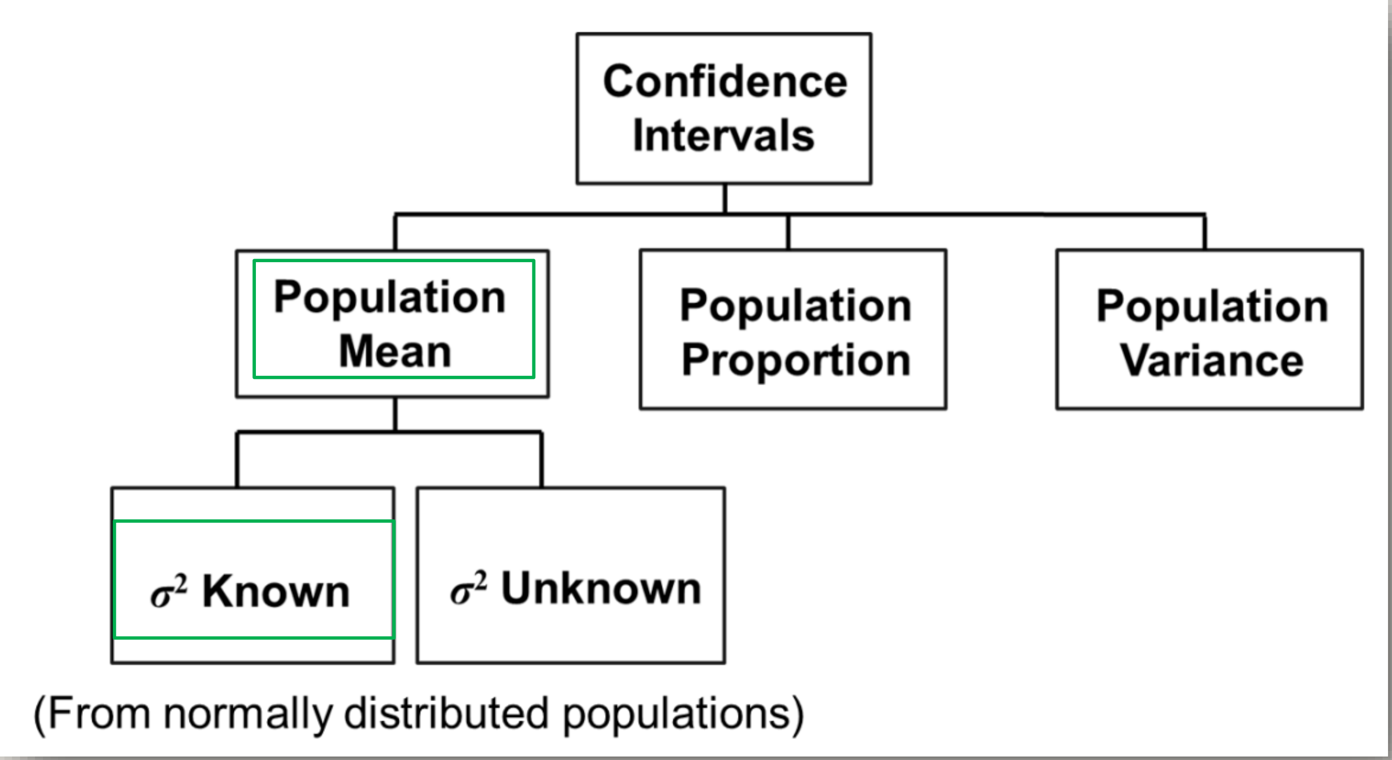
⇔ α = 0.04128

1- α = 1-0.04128= 0.95872 (95.87%)

Note:
The **margin of error** is half the width of the confidence interval.

LECTURE 10: ESTIMATORS AND CONFIDENCE INTERVALS

CONFIDENCE INTERVALS WE WILL CONSIDER



Newbold et al (2013)

CONFIDENCE INTERVAL ESTIMATE FOR THE MEAN (σ^2 KNOWN)

- Assumptions
  - Population variance σ^2 is known
  - Population is normally distributed
  - If population is not normal, use large sample
- Confidence interval estimate:

n ≥ 25

$$\boxed{\bar{x} \pm z_{1-\frac{\alpha}{2}} \times \frac{\sigma}{\sqrt{n}}}$$

($z_{1-\frac{\alpha}{2}}$ is the normal distribution value for a probability of $1-\frac{\alpha}{2}$ in each tail)

Quantile of the standard normal distribution corresponding to probability 1- α/2

Newbold et al (2013)

CI ESTIMATE FOR THE MEAN (σ^2 KNOWN): CONFIDENCE LIMITS

- The confidence interval is

$$\bar{x} \pm z_{1-\frac{\alpha}{2}} \times \frac{\sigma}{\sqrt{n}}$$

- The endpoints of the interval are

$$\text{UCL} = \bar{x} + z_{1-\frac{\alpha}{2}} \times \frac{\sigma}{\sqrt{n}}$$

Upper confidence limit

$$\text{LCL} = \bar{x} - z_{1-\frac{\alpha}{2}} \times \frac{\sigma}{\sqrt{n}}$$

Lower confidence limit

Newbold et al (2013)

CI ESTIMATE FOR THE MEAN (σ^2 KNOWN): MARGIN OF ERROR

- The confidence interval,

$$\bar{x} \pm z_{1-\frac{\alpha}{2}} \times \frac{\sigma}{\sqrt{n}}$$

- Can also be written as $\bar{x} \pm ME$

  where *ME* is called the margin of error

$$\boxed{ME = z_{1-\frac{\alpha}{2}} \times \frac{\sigma}{\sqrt{n}}}$$

- The interval width, *w*, is equal to twice the margin of error

Note:
The **margin of error** is half the width of the confidence interval. Equivalently, the **width of the interval** is twice the margin of error.

- Margin of error = (interval width) / 2
- Interval width = 2 × (margin of error)

Newbold et al (2013)

Width of Confidence Interval for the Mean (σ known)

$$\boxed{\text{Width} = 2 \cdot z_{1-\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}}$$

Where:

- $z_{1-\alpha/2}$ = critical value (quantile) from the standard normal distribution
- σ = population standard deviation
- n = sample size

CI ESTIMATE FOR THE MEAN (σ^2 KNOWN): REDUCING THE MARGIN OF ERROR

$$ME = z_{1-\frac{\alpha}{2}} \times \frac{\sigma}{\sqrt{n}}$$

The margin of error can be reduced if

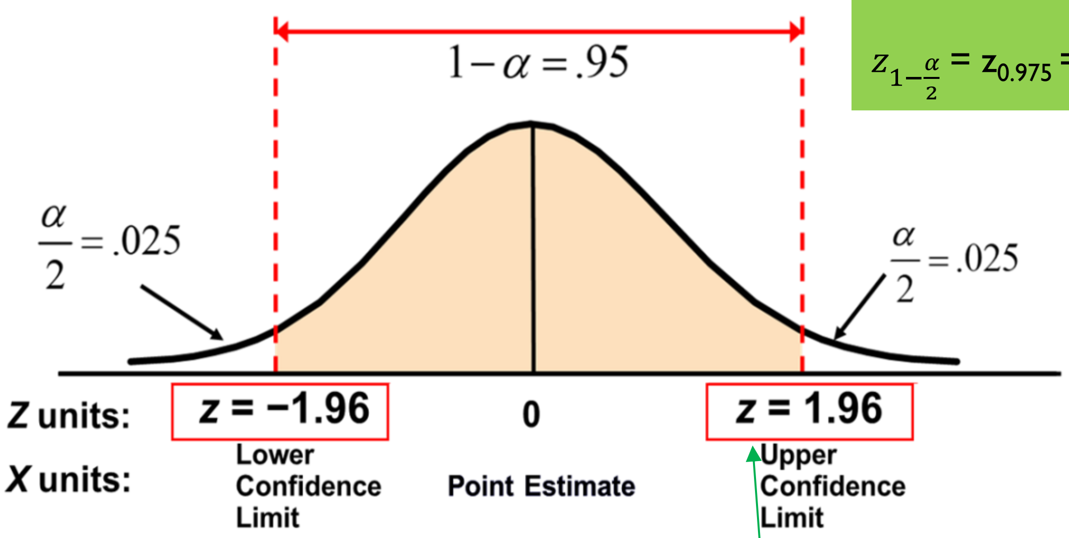
- the population standard deviation can be reduced $(\sigma \downarrow)$
- The sample size is increased $(n \uparrow)$
- The confidence level is decreased, $(1-\alpha)\downarrow$

Newbold et al (2013)

Note:
The **margin of error** decreases when the **standard deviation** is smaller or the **sample size** is larger, and increases when the **critical value** (confidence level) is higher.

Z-QUANTILES FOR A 95% CONFIDENCE INTERVAL

- Consider a 95% confidence interval:



- Find $Z_{0.975} = 1.96$ from the standard normal distribution table

$1-\alpha = 0.95 \Leftrightarrow \alpha = 0.05$

$1-\alpha/2 = 0.975$

$z_{1-\frac{\alpha}{2}} = z_{0.975} = 1.96$

Newbold et al (2013)

$z_{0.975} = 1.96$ is the quantile of the standard normal distribution corresponding to a cumulative probability of 0.975.

COMMON LEVELS OF CONFIDENCE

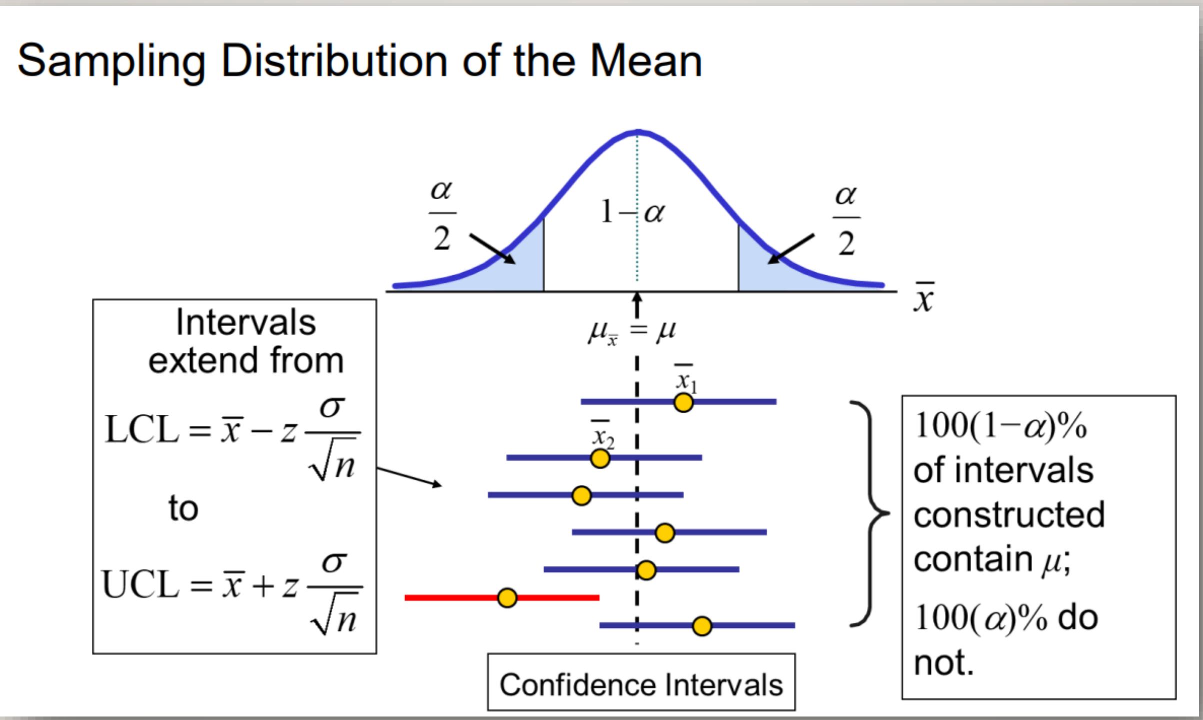
- Commonly used confidence levels are 90%, 95%, 98%, and 99%

Confidence Level	Confidence Coefficient, $1-\alpha$	Quantiles $z_{1-\frac{\alpha}{2}}$
80%	.80	1.28
90%	.90	1.645
95%	.95	1.96
98%	.98	2.33
99%	.99	2.58
99.8%	.998	3.08
99.9%	.999	3.27

Newbold et al (2013)

Note:
•The most commonly used confidence levels are **90%, 95%, and 99%**.

INTERVALS AND LEVEL OF CONFIDENCE



Newbold et al (2013)

Note:

•The **confidence level** is the probability that the interval, not the parameter, **contains the true population value**.

CI ESTIMATE FOR THE μ (σ^2 KNOWN): EXAMPLE

- A sample of 11 circuits from a large normal population has a mean resistance of 2.20 ohms. We know from past testing that the population standard deviation is 0.35 ohms.
- Determine a 95% confidence interval for the true mean resistance of the population.

Newbold et al (2013)

CI ESTIMATE FOR THE μ (σ^2 KNOWN): EXAMPLE

- A sample of 11 circuits from a large normal population has a mean resistance of 2.20 ohms. We know from past testing that the population standard deviation is .35 ohms.
- Solution:

$$\bar{x} \pm z_{1-\frac{\alpha}{2}} \times \frac{\sigma}{\sqrt{n}}$$

$$= 2.20 \pm 1.96\left(\frac{.35}{\sqrt{11}}\right)$$

$$= 2.20 \pm .2068$$

$$1.9932 < \mu < 2.4068$$

n = 11 (sample size)
$\bar{x}$ = 2.20 (sample mean)
σ = 0.35 (population standard deviation)
1 – α = 0.95 (confidence level), then α = 0.05 and $z_{1-\alpha/2} = z_{0.975} = 1.96$ (see previous slide)

Newbold et al (2013)

$CI_{0.95}(\mu) = (1.9932; 2.4068)$

- We are 95% confident that the true mean resistance is between 1.9932 and 2.4068 ohms
- Although the true mean may or may not be in this interval, 95% of intervals formed in this manner will contain the true mean

EXERCISE 7.14

7.14 It is known that the standard deviation in the volumes of 20-ounce (591-millliliter) bottles of natural spring water bottled by a particular company is 5 millliliters. One hundred bottles are randomly sampled and measured.

a. Calculate the standard error of the mean.
b. Find the margin of error of a 90% confidence interval estimate for the population mean volume.
c. Calculate the width for a 98% confidence interval for the population mean volume.

Newbold et al (2013)

EXERCISE 7.14 A): SOLUTION

Answer:

Given: $\sigma = 5$ ml, $n = 100$.

So the standard error is $\mathrm{SE} = \sigma/\sqrt{n} = 5/\sqrt{100} = 5/10 = 0.5$ ml.

(a) Standard error of the mean

$\bar{X}$ ~ Normal(μ, σ²/n)

$$\boxed{\mathrm{SE} = 0.5 \text{ ml}}$$

EXERCISE 7.14 B): SOLUTION

✅ Answer:

Given: $\sigma = 5$ ml, $n = 100$.

So the standard error is $\mathrm{SE} = \sigma/\sqrt{n} = 5/\sqrt{100} = 5/10 = 0.5$ ml.

$$\mathrm{ME} = z_{1-\frac{\alpha}{2}} \times \frac{\sigma}{\sqrt{n}}$$

1-α = 0.90 ⇔ α = 0.10

1- α/2 = 0.95

$z_{1-\frac{\alpha}{2}} = z_{0.95} = 1.645$

(b) Margin of error for a 90% confidence interval

For 90% two-sided CI, $Z_{0.95} = 1.645$

$$\mathrm{ME} = Z_{0.95} \cdot \mathrm{SE} = 1.645 \times 0.5 = 0.8225 \text{ ml}$$

Rounded sensibly:

$$\boxed{\mathrm{ME}_{90\%} \approx 0.823 \text{ ml}}$$

EXERCISE 7.14 C): SOLUTION

$$CI_{(1-\alpha)}(\mu) = \left(\bar{x} - z_{1-\frac{\alpha}{2}} \times \frac{\sigma}{\sqrt{n}}, \bar{x} + z_{1-\frac{\alpha}{2}} \times \frac{\sigma}{\sqrt{n}}\right)$$

Answer:

Given: $\sigma = 5$ ml, $n = 100$.

So the standard error is $\mathrm{SE} = \sigma/\sqrt{n} = 5/\sqrt{100} = 5/10 = 0.5$ ml.

(c) Width of a 98% confidence interval

For 98% two-sided CI, $Z_{0.99} = 2.3263$

$1\text{-}\alpha = 0.98 \Leftrightarrow \alpha = 0.02$

$1\text{-}\ \alpha/2 = 0.99$

$z_{1-\frac{\alpha}{2}} = z_{0.99} = 2.3263$

Margin of error:

$$\mathrm{ME}_{98\%} = 2.3263 \times 0.5 = 1.16315 \text{ ml}$$

Width of the interval (two-sided) = $2\times$ ME:

$$\mathrm{Width}_{98\%} = 2 \times 1.16315 = 2.3263 \text{ ml}$$

Rounded:

$$\boxed{\mathrm{Width}_{98\%} \approx 2.326 \text{ ml}}$$

Note:
The **margin of error** decreases when the **standard deviation** is smaller or the **sample size** is larger, and increases when the **critical value** (confidence level) is higher.

EXERCISE 7.15

7.15 A college admissions officer for an MBA program has determined that historically applicants have undergraduate grade point averages that are normally distributed with standard deviation 0.45. From a random sample of 25 applications from the current year, the sample mean grade point average is 2.90.

a. Find a 95% confidence interval for the population mean.
b. Based on these sample results, a statistician computes for the population mean a confidence interval extending from 2.81 to 2.99. Find the confidence level associated with this interval.

Newbold et al (2013)

EXERCISE 7.15 A): SOLUTION

Answer:

Problem data:

$\sigma = 0.45,\ n = 25,\ \bar{x} = 2.90.$

Because the population standard deviation σ is known and the underlying distribution is normal, use the normal (z) distribution.

(a) 95% confidence interval for μ

n = 25 (sample size)
$\bar{x}$ = 2.90 (sample mean)
σ = 0.45 (population standard deviation)
1 − α = 0.95 (confidence level), then
$z_{1-\frac{\alpha}{2}} = z_{0.975}$ = 1.96 (see previous slide)

Standard error:

$$SE = \frac{\sigma}{\sqrt{n}} = \frac{0.45}{\sqrt{25}} = \frac{0.45}{5} = 0.09.$$

Critical z for 95% (two-sided): $Z_{0.975} = 1.96$

$$CI_{(1-\alpha)}(\mu) = \left(\bar{x} - z_{1-\frac{\alpha}{2}} \times \frac{\sigma}{\sqrt{n}}, \bar{x} + z_{1-\frac{\alpha}{2}} \times \frac{\sigma}{\sqrt{n}}\right)$$

Margin of error:

$$ME = Z_{0.975} \times SE = 1.96 \times 0.09 = 0.1764.$$

Confidence interval:

$$\bar{x} \pm ME = 2.90 \pm 0.1764 \Rightarrow$$ $CI_{95\%}(\mu) = (2.7236, 3.0764)$

EXERCISE 7.15 B): SOLUTION

Answer:

Problem data:

$\sigma = 0.45,\ n = 25,\ \bar{x} = 2.90.$

Because the population standard deviation σ is known and the underlying distribution is normal, use the normal (z) distribution.

(b) Confidence level for the interval $CI_{1-\alpha}(\mu) = (2.81, 2.99)$

$$ME = z_{1-\frac{\alpha}{2}} \times \frac{\sigma}{\sqrt{n}} \Leftrightarrow$$

$$(2.99 - 2.81)/2 = z_{1-\frac{\alpha}{2}} \times 0.45/5 \Leftrightarrow$$

$$z_{1-\frac{\alpha}{2}} = 1, \text{ then } 1 - \alpha/2 = 0.8413 \Leftrightarrow \alpha = 0.3174$$

$$1 - \alpha = 0.6826$$

Standard Normal Distribution Table

n = 25 (sample size)
$\bar{x}$ = 2.90 (sample mean)
σ = 0.45 (population standard deviation)
1 – α = 0.95 (confidence level), then
$z_{1-\frac{\alpha}{2}} = z_{0.975}$ = 1.96 (see previous slide)

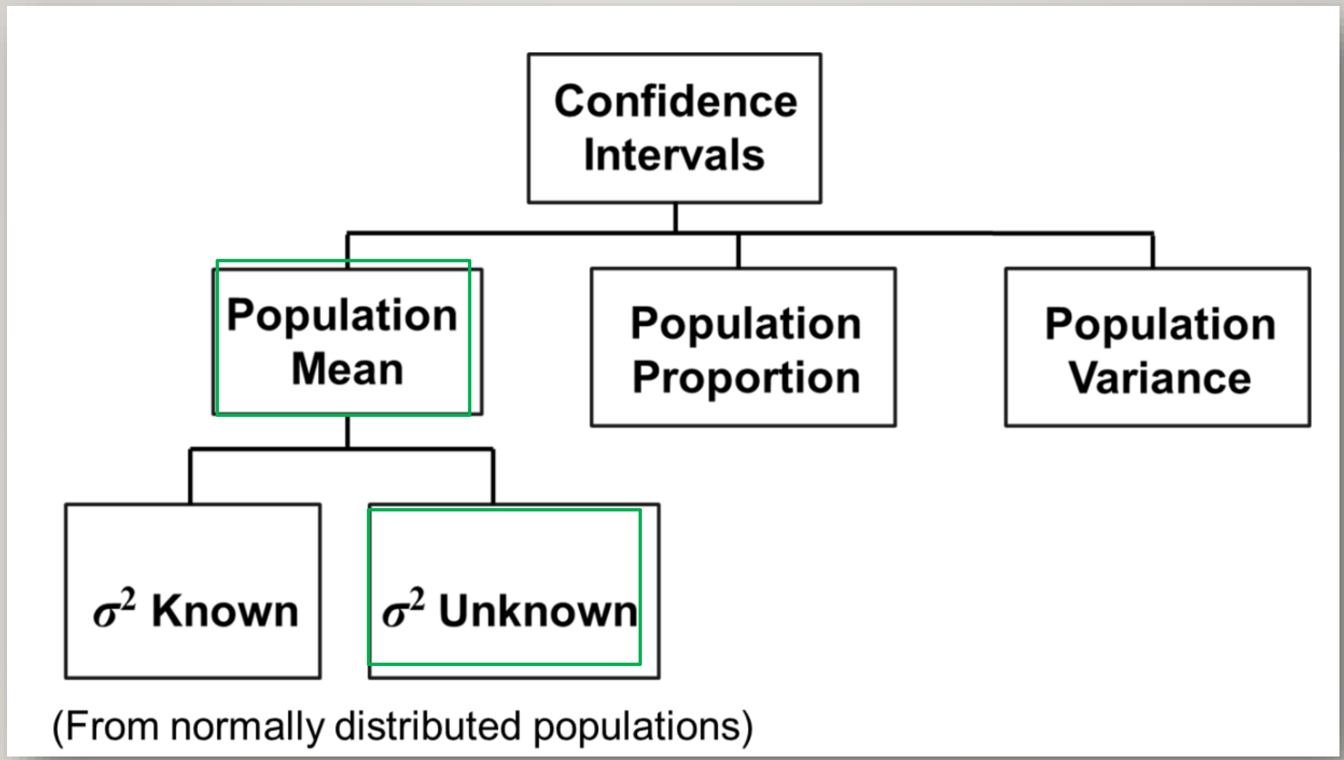
Thus, the confidence level is about 68.27%

Note:

- The **margin of error** is **half the width** of the confidence interval.

$$CI_{(1-\alpha)}(\mu) = \left(\bar{x} - z_{1-\frac{\alpha}{2}} \times \frac{\sigma}{\sqrt{n}}, \bar{x} + z_{1-\frac{\alpha}{2}} \times \frac{\sigma}{\sqrt{n}}\right)$$

CONFIDENCE INTERVAL ESTIMATION FOR THE MEAN (σ^2 UNKNOWN)



Newbold et al (2013)

CONFIDENCE INTERVAL ESTIMATION FOR THE MEAN (σ^2 UNKNOWN)

- If the population standard deviation σ is unknown, we can substitute the sample standard deviation, *s*
- This introduces extra uncertainty, since *s* is variable from sample to sample
- So we use the *t* distribution instead of the normal distribution

Newbold et al (2013)

CONFIDENCE INTERVAL ESTIMATION FOR THE MEAN (σ^2 UNKNOWN)

$$CI_{(1-\alpha)}(\mu) = \left(\bar{x} - t_{1-\frac{\alpha}{2};n-1} \times \frac{s}{\sqrt{n}}, \bar{x} + t_{1-\frac{\alpha}{2};n-1} \times \frac{s}{\sqrt{n}}\right)$$

- Assumptions
  - Population standard deviation is unknown
  - Population is normally distributed
  - If population is not normal, use large sample
- Use Student's *t* Distribution
- Confidence Interval Estimate:

$$\boxed{\bar{x} \pm t_{1-\frac{\alpha}{2};n-1} \times \frac{s}{\sqrt{n}}}$$

$t_{1-\frac{\alpha}{2};n-1}$ is the critical value of the *t* distribution with *n* − 1 d.f. and an area of $\frac{\alpha}{2}$ in each tail: $P\left(t_{n-1} > t_{1-\frac{\alpha}{2};n-1}\right) = \frac{\alpha}{2}$

Newbold et al (2013)

CI ESTIMATE FOR THE MEAN (σ^2 UNKNOWN): MARGIN OF ERROR

- The confidence interval,

$$CI_{(1-\alpha)}(\mu) = \left(\bar{x} - t_{1-\frac{\alpha}{2};n-1} \times \frac{s}{\sqrt{n}}, \bar{x} + t_{1-\frac{\alpha}{2};n-1} \times \frac{s}{\sqrt{n}}\right)$$

$$\boxed{\bar{x} \pm t_{1-\frac{\alpha}{2};n-1} \times \frac{s}{\sqrt{n}}}$$

- Can also be written as $\bar{x} \pm ME$

where *ME* is called the margin of error:

$$\boxed{\text{MSE} = t_{1-\frac{\alpha}{2};n-1} \times \frac{s}{\sqrt{n}}}$$

Newbold et al (2013)

CI ESTIMATE FOR THE μ (σ^2 UNKNOWN): EXAMPLE

A random sample of n = 25 has $\bar{x} = 50$ and s = 8. Form a 95% confidence interval for μ

- d.f. = n − 1 = 24, so $t_{1-\frac{\alpha}{2};n-1} = t_{0.975;24}$ = 2.0639

The confidence interval is

$$CI_{(1-\alpha)}(\mu) = \left(\bar{x} - t_{1-\frac{\alpha}{2};n-1} \times \frac{s}{\sqrt{n}}, \bar{x} + t_{1-\frac{\alpha}{2};n-1} \times \frac{s}{\sqrt{n}}\right)$$

$$\bar{x} \pm t_{1-\frac{\alpha}{2};n-1} \times \frac{s}{\sqrt{n}}$$

n = 25 (sample size)
$\bar{x} = 50$ (sample mean)
s = 8 (sample standard deviation)
1 – α = 0.95 (confidence level), then
$t_{0.975;24}$ = 2.0639 (see Student´s table)

$$50 \pm (2.0639) \frac{8}{\sqrt{25}}$$

$$46.698 < \mu < 53.302$$

$CI_{95\%}$ (μ) = (46.698, 53.302)

Newbold et al (2013)

EXERCISE 7.28

7.28 A business school placement director wants to estimate the mean annual salaries 5 years after students graduate. A random sample of 25 such graduates found a sample mean of $42,740 and a sample standard deviation of $4,780. Find a 90% confidence interval for the population mean, assuming that the population distribution is normal.

Newbold et al (2013)

EXERCISE 7.28: SOLUTION

Answer:

Given data

$$n = 25, \quad \bar{x} = 42{,}740, \quad s = 4{,}780$$

Because the **population standard deviation** is *unknown* and the **sample size is small (n = 25)**, we use the **t-distribution** with

$$df = n - 1 = 24$$

n = 25 (sample size)
$\bar{x} = 42.740$ (sample mean)
s = 4.780 (sample standard deviation)
$1 - \alpha = 0.90$ (confidence level),
then $t_{1-\alpha/2;\, n-1} = t_{0.95;24} = 1.711$ (see Student´s table)

Compute the standard error

$$SE = \frac{s}{\sqrt{n}} = \frac{4780}{\sqrt{25}} = \frac{4780}{5} = 956$$

Find the critical t-value for a 90% confidence interval

m the t-table (or calculator):

$$t_{0.95;24} = 1.711$$

Student´s t Distribution Table

EXERCISE 7.28: SOLUTION

Answer:

Given data

$$n = 25, \quad \bar{x} = 42{,}740, \quad s = 4{,}780$$

Because the **population standard deviation** is *unknown* and the **sample size is small (n = 25)**, we use the **t-distribution** with

$$df = n - 1 = 24$$

n = 25 (sample size)
$\bar{x} = 42.740$ (sample mean)
s = 4.780 (sample standard deviation)
1 – α = 0.90 (confidence level),
then $t_{1-\alpha/2;\, n-1} = t_{0.95;24} = 1.711$ (see Student´s table)

Compute the margin of error

$$ME = t_{0.95;24} \times SE = 1.711 \times SE = 1.711 \times 956 = 1636.82$$

Construct the confidence interval

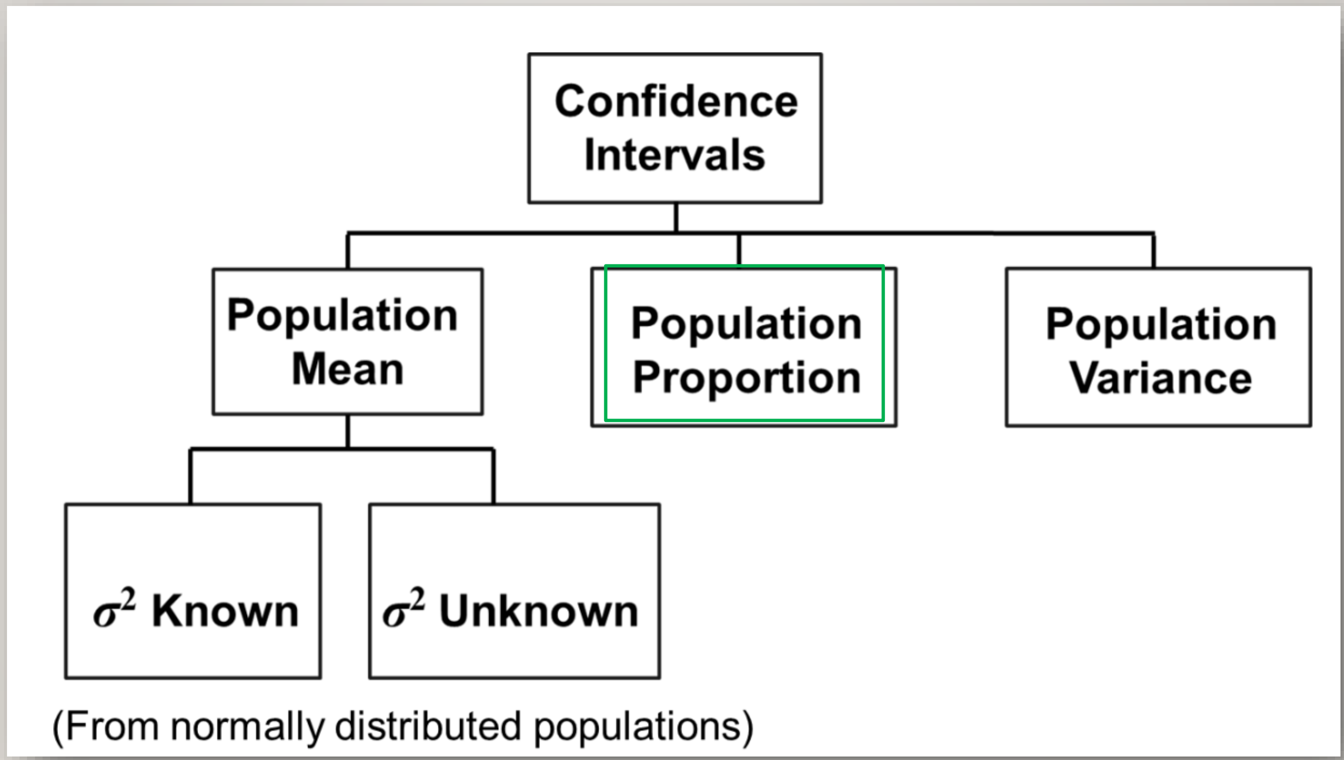
$$\bar{x} \pm ME = 42{,}740 \pm 1{,}637$$

$$\boxed{90\%\ CI : [41{,}103,\ 44{,}377]}$$

$$CI_{(1-\alpha)}(\mu) = \left(\bar{x} - t_{1-\frac{\alpha}{2};n-1} \times \frac{s}{\sqrt{n}}, \bar{x} + t_{1-\frac{\alpha}{2};n-1} \times \frac{s}{\sqrt{n}}\right)$$

$CI_{90\%}(\mu) = (41.103, 44.377)$

CONFIDENCE INTERVAL ESTIMATION FOR POPULATION PROPORTION



Newbold et al (2013)

CONFIDENCE INTERVAL ESTIMATION FOR POPULATION PROPORTION

- An interval estimate for the population proportion (P) can be calculated by adding an allowance for uncertainty to the sample proportion $(\hat{p})$

Newbold et al (2013)

CONFIDENCE INTERVALS FOR THE POPULATION PROPORTION

- Recall that the distribution of the sample proportion is approximately normal if the sample size is large, with standard deviation

$$\sigma_P = \sqrt{\frac{P(1-P)}{n}}$$

$$Z = \frac{\hat{p} - P}{\sqrt{\frac{P(1-P)}{n}}} \sim \text{Normal(0, 1)}$$

- We will estimate this with sample data:

$$\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

Newbold et al (2013)

CONFIDENCE INTERVAL ENDPOINTS

- The confidence interval for the population proportion is given by

$$\boxed{\hat{p} \pm z_{1-\frac{\alpha}{2}} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}}$$

- where

  $z_{1-\frac{\alpha}{2}}$ is the standard normal value for the level of confidence desired

  - $\hat{p}$ is the sample proportion
  - n is the sample size
  - n ≥ 25

Newbold et al (2013)

$$CI_{(1-\alpha)}(p) = \left(\hat{p} - z_{1-\frac{\alpha}{2}} \times \sqrt{\frac{\hat{p}(\mathbf{1}-\hat{p})}{\boldsymbol{n}}}, \hat{p} + z_{1-\frac{\alpha}{2}} \times \sqrt{\frac{\hat{p}(\mathbf{1}-\hat{p})}{\boldsymbol{n}}}\right)$$

CONFIDENCE INTERVALS FOR THE POPULATION PROPORTION: EXAMPLE

- A random sample of 100 people shows that 25 are left-handed.
- Form a 95% confidence interval for the true proportion of left-handers

Newbold et al (2013)

CONFIDENCE INTERVALS FOR THE POPULATION PROPORTION: EXAMPLE

- A random sample of 100 people shows that 25 are left-handed. Form a 95% confidence interval for the true proportion of left-handers.

n = 100 (sample size)
$\hat{p} = 25/100$ (sample proportion)
1 – α = 0.95 (confidence level), then $z_{1-\frac{\alpha}{2}} = z_{0.975}$ = 1.96 (see Standard Normal Distribution Table)

$$\hat{p} \pm z_{1-\frac{\alpha}{2}} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

$$\frac{25}{100} \pm 1.96 \sqrt{\frac{.25(.75)}{100}}$$

$$0.1651 < P < 0.3349$$

$CI_{95\%}$ (p) = (0.1651, 0.3349)

$CI_{95\%}$ (p) = (16.51%, 33.49%)

Newbold et al (2013)

Interpretation:
- We are 95% confident that the true proportion of left-handers in the population is between 16.51% and 33.49%.
- Although the interval from 0.1651 to 0.3349 may or may not contain the true proportion, 95% of intervals formed from samples of size 100 in this manner will contain the true proportion.

EXERCISE 7.40

7.40 Suppose that the local authorities in a heavily populated residential area of downtown Hong Kong were considering building a new municipal swimming pool and leisure center. Because such a development would cost a great deal of money, it first of all needed to be established whether the residents of this area thought that the swimming pool and leisure center would be a worthwhile use of public funds. If 243 out of a random sample of 360 residents in the local area thought that the pool and leisure center should be built, determine with 95% confidence the proportion of all the local residents in the area who would support the proposal.

Newbold et al (2013)

EXERCISE 7.40: SOLUTION

$$CI_{(1-\alpha)}(p) = \left(\hat{p} - z_{1-\frac{\alpha}{2}} \times \sqrt{\frac{\hat{p}(\mathbf{1}-\hat{p})}{\boldsymbol{n}}}, \hat{p} + z_{1-\frac{\alpha}{2}} \times \sqrt{\frac{\hat{p}(\mathbf{1}-\hat{p})}{\boldsymbol{n}}}\right)$$

Answer:

Sample information:

- Sample size $n = 360$
- Number who support the proposal $x = 243$

1. Sample proportion

$$\hat{p} = \frac{x}{n} = \frac{243}{360} = 0.675$$

2. Standard error

$$SE = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\frac{(0.675)(0.325)}{360}} = \sqrt{0.000609375} \approx 0.0247$$

3. 95% Confidence Interval

Use $z = 1.96$ for 95% confidence:

$$\hat{p} \pm 1.96 \cdot SE = 0.675 \pm 1.96(0.0247)$$

$$= 0.675 \pm 0.0484$$

$$\text{Lower limit} \approx 0.6266 \quad \text{and} \quad \text{Upper limit} \approx 0.7234$$

n = 300 (sample size)
$\hat{p} = 243/360$ (sample proportion)
1 – α = 0.95 (confidence level),
then $z_{1-\frac{\alpha}{2}} = z_{0.975} = 1.96$ (see Standard Normal Distribution Table)

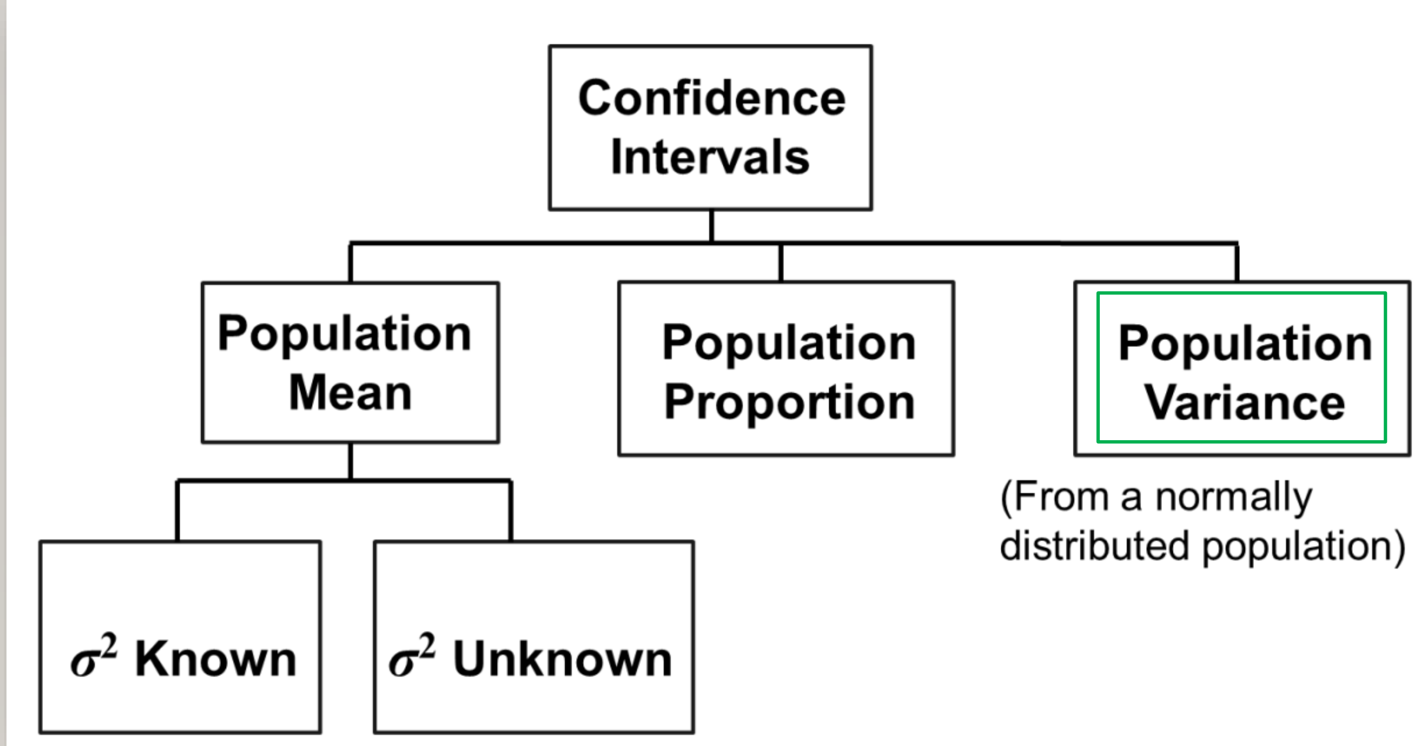
Final Answer (95% CI):

$$\boxed{0.63 \le p \le 0.72} \quad \text{(approximately)}$$

$CI_{95\%}$ (p) = (0.63, 0.72)

So, we are 95% confident that **between 63% and 72%** of all local residents would support building the swimming pool and leisure center.

ESTIMATION FOR THE POPULATION VARIANCE



Newbold et al (2013)

CONFIDENCE INTERVALS FOR THE POPULATION VARIANCE

- Goal: Form a confidence interval for the population variance, σ^2
  - The confidence interval is based on the sample variance, s^2
  - Assumed: the population is normally distributed

Newbold et al (2013)

CONFIDENCE INTERVALS FOR THE POPULATION VARIANCE

The random variable

$$Q = \frac{(n-1)s^2}{\sigma^2}$$

$$\mathbf{Q} = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{(n-1)}$$

follows a chi-square distribution with (n − 1) degrees of freedom

Where the chi-square value $\chi^2_{n-1,1-\alpha}$ is the number for which

$$P\left(\chi^2_{n-1} > \chi^2_{n-1,1-\alpha}\right) = \alpha$$

Newbold et al (2013)

CONFIDENCE INTERVALS FOR THE POPULATION VARIANCE

The $100(1-\alpha)\%$ confidence interval for the population variance is given by

$$\text{LCL} = \frac{(n-1)s^2}{\chi^2_{(1-\alpha/2,\ n-1)}}$$

$$\text{UCL} = \frac{(n-1)s^2}{\chi^2_{(\alpha/2,\ n-1)}}$$

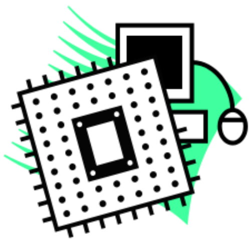
Newbold et al (2013)

$$CI_{(1-\alpha)}(\sigma^2) = \left(\frac{(n-1)s^2}{\chi^2_{1-\frac{\alpha}{2},n-1}}, \frac{(n-1)s^2}{\chi^2_{\frac{\alpha}{2},n-1}} \right)$$

CONFIDENCE INTERVALS FOR THE POPULATION VARIANCE: EXAMPLE

You are testing the speed of a batch of computer processors. You collect the following data (in Mhz):

Sample size	**17**
Sample mean	**3004**
Sample std dev	**74**



Assume the population is normal.

Determine the 95% confidence interval for σ_x^2

Newbold et al (2013)

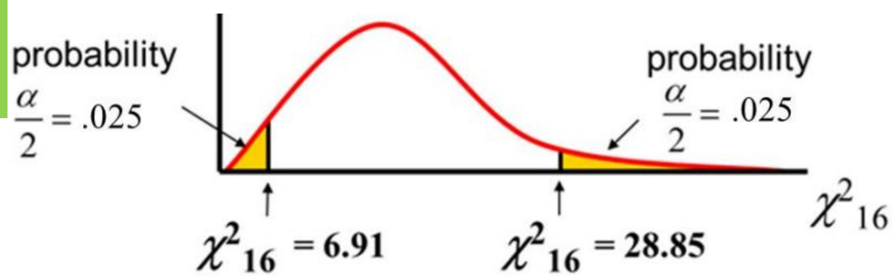
FINDING THE CHI-SQUARE VALUES

- n = 17 so the chi-square distribution has (n − 1) = 16 degrees of freedom
- $\alpha = 0.05$, so use the the chi-square values with area 0.025 in each tail:

n = 17 (sample size)
s = 74 (sample standard deviation)
α = 0.05 (significance level)
1 – α = 0.95 (confidence level)
$\chi^2_{(0.025,\,16)}$ = 6.91
$\chi^2_{(0.975,\,16)}$ = 28.85 (see Chi-Square Distribution Table)

$$\chi^2_{(1-\alpha/2,\,n-1)} = \chi^2_{(0.975,\,16)} = 28.85$$

$$\chi^2_{(\alpha/2,\,n-1)} = \chi^2_{(0.025,\,16)} = 6.91$$



Newbold et al (2013)

CALCULATING THE CONFIDENCE LIMITS

- The 95% confidence interval is

$$CI_{(1-\alpha)}(\sigma^2) = \left(\frac{(n-1)s^2}{\chi^2_{1-\frac{\alpha}{2},n-1}}, \frac{(n-1)s^2}{\chi^2_{\frac{\alpha}{2},n-1}}\right)$$

$$\frac{(n-1)s^2}{\chi^2_{(1-\alpha/2,\, n-1)}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{(\alpha/2,\, n-1)}}$$

$$\frac{(17-1)(74)^2}{28.85} < \sigma^2 < \frac{(17-1)(74)^2}{6.91}$$

$$3037 < \sigma^2 < 12680$$

$CI_{95\%}(\sigma^2) = (3037, 12680)$

Converting to standard deviation, we are 95% confident that the population standard deviation of CPU speed is between 55.1 and 112.6 Mhz

Newbold et al (2013)

$CI_{95\%}(\sigma) = (55.1, 112.6)$

EXERCISE 7.49

7.49 A manufacturer is concerned about the variability of the levels of impurity contained in consignments of raw material from a supplier. A random sample of 15 consignments showed a standard deviation of 2.36 in the concentration of impurity levels. Assume normality.

a. Find a 95% confidence interval for the population variance.
b. Would a 99% confidence interval for this variance be wider or narrower than that found in part a?

Newbold et al (2013)

EXERCISE 7.49 A): SOLUTION

Answer:

We use the chi-square confidence interval for the population **variance**.

Given $n = 15$, sample standard deviation $s = 2.36$, so $s^2 = 5.5696$. Degrees of freedom $\nu = n - 1 = 14$.

The $100(1-\alpha)\%$ CI for the variance σ^2 is

$$\left(\frac{(\nu)s^2}{\chi^2_{1-\alpha/2,\nu}}, \frac{(\nu)s^2}{\chi^2_{\alpha/2,\nu}} \right).$$

$$CI_{(1-\alpha)}(\sigma^2) = \left(\frac{(n-1)s^2}{\chi^2_{1-\frac{\alpha}{2},n-1}}, \frac{(n-1)s^2}{\chi^2_{\frac{\alpha}{2},n-1}} \right)$$

a) 95% confidence interval ($\alpha = 0.05$)

For $\nu = 14$:

$$\chi^2_{0.975,14} \approx 26.11895, \qquad \chi^2_{0.025,14} \approx 5.62873.$$

Compute numerator $(\nu)s^2 = 14 \times 5.5696 = 77.9744$.

Thus

$$\text{Lower bound for } \sigma^2 = \frac{77.9744}{26.11895} \approx 2.9854,$$

$$\text{Upper bound for } \sigma^2 = \frac{77.9744}{5.62873} \approx 13.8529.$$

$CI_{95\%}(\sigma^2) = (2.99, 13.85)$

So a 95% CI for the **variance** is approximately

$$\boxed{2.99 \le \sigma^2 \le 13.85}.$$

For interpretation it is common to also give the CI for the **standard deviation** by taking square roots:

$$\sqrt{2.9854} \approx 1.728, \qquad \sqrt{13.8529} \approx 3.722,$$

so a 95% CI for σ is approximately $\boxed{1.73 \le \sigma \le 3.72}$.

$CI_{95\%}(\sigma) = (1.73, 3.72)$

EXERCISE 7.49 B): SOLUTION

Answer:

We use the chi-square confidence interval for the population **variance**.

Given $n = 15$, sample standard deviation $s = 2.36$, so $s^2 = 5.5696$. Degrees of freedom $\nu = n - 1 = 14$.

The $100(1-\alpha)\%$ CI for the variance σ^2 is

$$\left(\frac{(\nu)s^2}{\chi^2_{1-\alpha/2,\nu}}, \frac{(\nu)s^2}{\chi^2_{\alpha/2,\nu}} \right).$$

b) Would a 99% CI be wider or narrower?

A 99% confidence interval would be **wider**. (Greater confidence requires a larger interval: the chi-square quantiles move farther into the tails, which increases the spread of the interval.)

(For reference, the 99% CI for σ^2 would be about $(2.49,\ 19.14)$, confirming it is wider than the 95% interval.)

$CI_{99\%}(\sigma^2) = (2.49, 19.14)$

LECTURE 10: HYPOTHESIS TESTS

WHAT IS A HYPOTHESIS TEST?

- A **hypothesis test** is a statistical procedure used to make a decision or draw a conclusion about a population parameter, based on sample data.
  It evaluates whether there is enough evidence to reject a claim (the **null hypothesis**) in favor of an alternative claim (the **alternative hypothesis**), using a test statistic and a significance level.

A hypothesis is an assumption about the population parameter (say population mean) which is to be tested.

For that we collect sample data , then we calculate sample statistics (say sample mean) and then use this information to judge/decide whether hypothesized value of population parameter is correct or not.

CONSTRUCTION OF A HYPOTHESIS TEST

- Objectives and Data
- Conditions of Applicability
- Hipotheses and Significance Level
- Test Statistic
- P-value / Critical Value
- Decision and Conclusion

WHAT IS A HYPOTHESIS?

A hypothesis is a statement about a population parameter that can be tested using sample data.

Example: The mean weight of this class is 58 kg?

Null and Alternative Hypotheses:

- The **null hypothesis (H_0)** is a statement of no effect or no difference. It represents the default or status quo.
- The **alternative hypothesis (H_1 or H_a)** is a statement that contradicts the null, representing the effect or difference we want to detect.

Examples for the population mean (μ):

Type of test	Null hypothesis (H_0)	Alternative hypothesis (H_1)
Two-tailed	$\mu = \mu_0$	$\mu \neq \mu_0$
Right-tailed	$\mu \leq \mu_0$	$\mu > \mu_0$
Left-tailed	$\mu \geq \mu_0$	$\mu < \mu_0$

Note:
Different null and alternative hypotheses can be considered depending on the parameter of interest, and the **type of test** is determined accordingly.

TYPES OF ERRORS VS. SIGNIFICANCE LEVEL

Types of Errors

α= P (Type I error) = P (Reject Ho/Ho True)
β = P (Type II error) = P (accept Ho/ Ho false)

Type I error (α): Rejecting H_0 when H_0 is true.
Type II error (β): Failing to reject H_0 when H_0 is false.

Power of a Test

Condition	Reject H_0	Accept H_0
H_0 is false	Correct decision $1-\beta$	Incorrect decision (Typo II error) β
H_0 is true	Incorrect decision (Type I error) α	Correct decision $1-\alpha$

Power of a Test:
The power of a test is the probability of correctly rejecting the null hypothesis (H_0) when it is false.
Mathematically, it is given by:
Power $= 1 - \beta$
where β is the probability of a Type II error.

Significance Level (0 ≤ α ≤ 1)

- The probability that the researcher sets a priori as the threshold to decide whether to reject H_0.
- Common significance levels: 1%, 5%, and 10%

REJECTION REGION VS. P-VALUE

Rejection Region

- The rejection region is also called the critical region , is the range of sample statistics values within which if values of sample statistics falls , then Ho rejected.
- It is outside the limit of acceptance region.
- The critical value is the cut off value of the sample statistics which acts as a boundary and separates the regions of acceptance or rejections

Rejection Region (RR) or Critical Region (CR):
The set of values for which H_0 is rejected

- Left-Tailed Test: RR = $]-\infty; z_\alpha]$
- Right-Tailed Test: RR = $[z_{1-\alpha}; +\infty[$
- Two-Tailed Test : RR = $]-\infty; -z_{1-\alpha/2}] \cup [z_{1-\alpha/2}; +\infty[$

"Decision Rule (using critical values):

- $z_0 \leq z_\alpha \Rightarrow$ Reject H_0
- $z_0 \geq z_{1-\alpha} \Rightarrow$ Reject H_0
- $|z_0| \geq z_{1-\alpha/2} \Rightarrow$ Reject H_0"

"Rule: $z_0 \in RR \Rightarrow$ Reject H_0"

Note:
If the value of the test statistic falls within the rejection region, then we reject H_0 at the chosen significance level.

P-value: is the probability of obtaining a test statistic at least as extreme as the one observed, assuming that hull hypotesis H_0 is true.

- Left-Tailed Test: P-value = $P(Z \leq z_0)$
- Right-Tailed Test: P-value = $P(Z \geq z_0)$
- Two-Tailed Test: P-value = $P(Z \leq -z_0$ ou $Z \geq z_0) = 2 \times P(Z \geq |z_0|)$

"Rule: P-value < $\alpha \Rightarrow$ Reject H_0"

Note:
The **P-value** helps to determine whether the observed data are consistent with H_0:

- A **small p-value** (typically $\leq \alpha$) indicates strong evidence against H_0, so we reject H_0.
- A **large p-value** ($> \alpha$) indicates weak evidence against H_0, so we fail to reject H_0.

Note:
If the p-value is smaller than the chosen significance level (α), then we reject H_0 at that significance level.

TESTS OF THE MEAN OF A NORMAL DISTRIBUTION (σ^2 KNOWN)

1. Hypotheses

- Null hypothesis: $H_0 : \mu = \mu_0$, H_0: $\mu \le \mu_0$ OR H_0: $\mu \ge \mu_0$
- Alternative hypothesis: $H_1 : \mu \neq \mu_0$ (two-tailed)

  or $H_1 : \mu > \mu_0$ / $H_1 : \mu < \mu_0$ (one-tailed)

2. Test Statistic

$$\boxed{Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \sim \mathrm{Normal}(0,1)}$$

where $\bar{X}$ is the sample mean, σ the population standard deviation, and n the sample size.

3. Decision Rule

- Using critical value(s): Reject H_0 if $Z \in \mathrm{Rejection\ Region}$
- Using p-value: Reject H_0 if $p\text{-value} < \alpha$

TESTS OF THE MEAN OF A NORMAL DISTRIBUTION (σ^2 KNOWN)

- A parametric hypothesis test for the parameter μ (the population mean) may be:

Two-Tailed Test

$H_0: \mu = \mu_0$

$H_1: \mu \neq \mu_0$

Right-Tailed Test

$H_0: \mu \leq \mu_0$

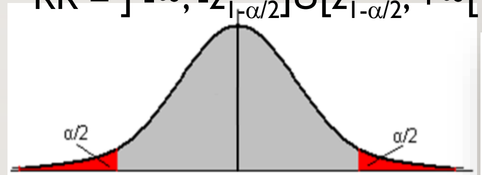
$H_1: \mu > \mu_0$

Left-Tailed Test

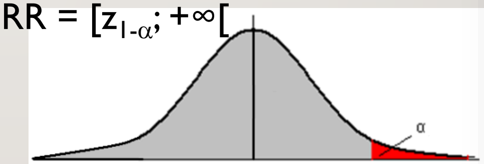
$H_0: \mu \geq \mu_0$

$H_1: \mu < \mu_0$

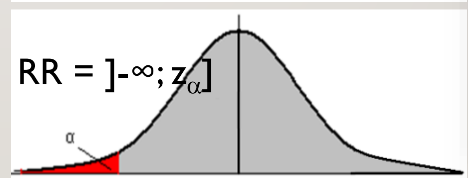
RR = $]-\infty; -z_{1-\alpha/2}] \cup [z_{1-\alpha/2}; +\infty[$



RR = $[z_{1-\alpha}; +\infty[$



RR = $]-\infty; z_{\alpha}]$



Two-tailed test:

- Rejection Region: $|Z| \geq Z_{1-\alpha/2}$
- P-value: $2 \cdot P(Z \geq |z_0|)$

Right-tailed test:

- Rejection Region: $Z \geq Z_{1-\alpha}$
- P-value: $P(Z \geq z_0)$

Left-tailed test:

- Rejection Region: $Z \leq Z_{\alpha}$
- P-value: $P(Z \leq z_0)$

where μ_0 is the specific numerical value considered in H_0 and H_1.

EXERCISE 9.11

9.11 A manufacturer of detergent claims that the contents of boxes sold weigh on average at least 16 ounces. The distribution of weight is known to be normal, with a standard deviation of 0.4 ounce. A random sample of 16 boxes yielded a sample mean weight of 15.84 ounces. Test at the 10% significance level the null hypothesis that the population mean weight is at least 16 ounces.

Newbold et al (2013)

EXERCISE 9.11: SOLUTION

Answer:

Step 1: State the hypotheses

$$H_0 : \mu \geq 16 \quad \text{(the mean weight is at least 16 oz)}$$

$$H_1 : \mu < 16 \quad \text{(the mean weight is less than 16 oz)}$$

This is a **left-tailed** test.

Left-tailed Test

Step 2: Given information

$$\sigma = 0.4, \quad n = 16, \quad \bar{x} = 15.84, \quad \alpha = 0.10$$

EXERCISE 9.11: SOLUTION

Answer:

Step 3: Test Statistic

Since the population standard deviation is known and the population is normal, we use a **Z-test**:

$$Z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{15.84 - 16}{0.4/\sqrt{16}}$$

$$Z = \frac{-0.16}{0.1} = -1.6$$

Step 4: Critical Value / **p-value**

Left-Tailed Test: RR =]-∞; z_{α}] =]-∞; -1.282]

- Left-tailed test, α = 0.10 → $z_{0.10} \approx -1.282$

$$p\text{-value} = P(Z < -1.6) \approx 0.055$$

Standard Normal Distribution Table

EXERCISE 9.11: SOLUTION

Answer:

Step 5: Decision Rule

Compare the test statistic to the critical value:

$$Z = -1.6 \quad \text{is less than} \quad -1.28$$

- Calculated z = -1.6 < -1.282 → **reject** H_0
- p-value ≈ 0.055 < 0.10 → **reject** H_0

$-1.6 \in RR =]-\infty; z_\alpha] =]-\infty; -1.282]$

Conclusion

There is **sufficient evidence at the 10% significance level** to conclude that the mean weight of the detergent boxes is **less than 16 ounces**. The manufacturer's claim is **not supported**.

Note:
In the following slides, we will examine **both the right-tailed and two-tailed tests** to compare the results.

EXERCISE 9.11: TYPES OF HYPOTHESIS TESTS FOR THE MEAN (σ^2 KNOWN)

Answer:

1 Left-tailed

- $H_0 : \mu \geq 16$
- $H_1 : \mu < 16$

> Test whether the mean weight is less than 16 ounces.

2 Right-tailed

- $H_0 : \mu \leq 16$
- $H_1 : \mu > 16$

> Test whether the mean weight is greater than 16 ounces.

3 Two-tailed

- $H_0 : \mu = 16$
- $H_1 : \mu \neq 16$

> Test whether the mean weight differs from 16 ounces.

EXERCISE 9.11: RIGHT-TAILED TEST

Answer:

Problem Setup:

- Population standard deviation: σ = 0.4
- Sample size: n = 16
- Sample mean: $\bar{x} = 15.84$
- Hypotheses:

$$H_0 : \mu \leq 16 \quad , \quad H_1 : \mu > 16$$

- Significance level: α = 0.10

Right-tailed Test

Test statistic (z):

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{15.84 - 16}{0.4/\sqrt{16}} = \frac{-0.16}{0.1} = -1.6$$

Critical value: $z_{0.9} = 1.282$

Decision rule: Reje

Right-Tailed Test: RR = [$z_{1-\alpha}$; +∞[= [1.282, +∞[

Conclusion:

$$p\text{-value} = P(Z > -1.6) = 1 - P(Z < -1.6) \approx 1 - 0.0548 = 0.9452$$

- **Critical value:** z = -1.6 < 1.282 → do not reject H_0
- **p-value:** 0.945 > 0.10 → do not reject H_0
- **Interpretation:** Not enough evidence to conclude the mean weight is **greater than 16** ounces.

$-1.6 \notin$ RR = [1.282, +∞[

EXERCISE 9.11: TWO-TAILED TEST

Answer:

Problem Setup:

- Same data as above
- Hypotheses:

Two-tailed Test

$$H_0 : \mu = 16 \quad , \quad H_1 : \mu \neq 16$$

- Significance level: α = 0.10

Test statistic (z):

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = -1.6$$

Critical values:

- Two-tailed: $z_{0.05} = \pm 1.645$

Two-Tailed Test : RR =] -∞; $-z_{1-\alpha/2}$]U[$z_{1-\alpha/2}$; +∞[
=] -∞; -1.645]U[1.645; +∞[

Decision

$$p\text{-value} = 2 \cdot P(Z < -1.6) \approx 2 \cdot 0.0548 = 0.1096$$

Conclusion:

- **Critical value:** |z| = 1.6 < 1.645 → do not reject H_0 ⟺ -1.6 ∉ RR =] -∞; -1.645]U[1.645; +∞[
- **p-value:** 0.110 > 0.10 → do not reject H_0
- **Interpretation:** Not enough evidence to conclude the mean weight **differs from 16 ounces.**

HOMEWORK OF LECTURE 10: QUESTIONS

EXERCISE 7.29

7.29 A car-rental company is interested in the amount of time its vehicles are out of operation for repair work. State all assumptions and find a 90% confidence interval for the mean number of days in a year that all vehicles in the company's fleet are out of operation if a random sample of nine cars showed the following number of days that each had been inoperative:

16 10 21 22 8 17 19 14 19

Newbold et al (2013)

EXERCISE 7.41

7.41 It is important for airlines to follow the published scheduled departure times of flights. Suppose that one airline that recently sampled the records of 246 flights originating in Orlando found that 10 flights were delayed for severe weather, 4 flights were delayed for maintenance concerns, and all the other flights were on time.

a. Estimate the percentage of on-time departures using a 98% confidence level.
b. Estimate the percentage of flights delayed for severe weather using a 98% confidence level.

Newbold et al (2013)

EXERCISE 7.50

7.50 A manufacturer bonds a plastic coating to a metal surface. A random sample of nine observations on the thickness of this coating is taken from a week's output, and the thicknesses (in millimeters) of these observations are as follows:

19.8 21.2 18.6 20.4 21.6 19.8 19.9 20.3 20.8

Assuming normality, find a 90% confidence interval for the population variance.

Newbold et al (2013)

THANKS!

Questions?