STATISTICAL LABORATORY

Applied Mathematics for Economics and Management
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https://doity.com.br/estatistica-aplicada-a-nutricao

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PROGRAM

1. Fundamental Concepts of Statistics
2. Exploratory Data Analysis
3. Organizing and Summarizing Data
4. Association and Relationships Between Variables
5. Index Numbers
6. Time Series Analysis

LECTURE 10: TREND ANALYSIS OF A TIME SERIES

TREND ANALYSIS OF A TIME SERIES

In this section, we study the **trend component** of a time series using two methods:

1. **Analytical Method**
2. **Moving Average Method**

We assume that the series does **not** present **seasonality**, and that the cyclical component is not being considered.
Therefore, the model is:

$$\boxed{x_t = T_t + e_t}$$

where:
x_t = **observed value of the time series at time** t
T_t = **trend**
e_t = **irregular/random component**
$t = 1,2, \dots , n$

ANALYTICAL METHOD – LINEAR TREND

We assume that the time series follows a **linear trend**:

$$\boxed{x_t = T_t + e_t = a + bt + e_t}$$

where:

x_t = **observed value at time** t

T_t = **trend component**

e_t = **irregular/random component (random error)**

$t = 1,2, \ldots, n$

ANALYTICAL METHOD – LINEAR TREND

Estimation of Parameters: Least Squares Method

The parameters a and b are determined using the **Least Squares Method**. Thus, the parameters a and b are chosen to **minimize the sum of squared residuals**:

$$\boxed{\sum_{t=1}^{n} e_t^2 = \sum_{t=1}^{n} (x_t - a - bt)^2}$$

Differentiating and solving gives the estimators:

$$\boxed{b = \frac{n\sum_{t=1}^{n} t x_t - \left(\sum_{t=1}^{n} t\right)\left(\sum_{t=1}^{n} x_t\right)}{n\sum_{t=1}^{n} t^2 - \left(\sum_{t=1}^{n} t\right)^2}} \quad \text{and} \quad \boxed{a = \bar{x} - b\bar{t}}$$

where $\bar{x} = \frac{1}{n}\sum_{t=1}^{n} x_t$ and $\bar{t} = \frac{1}{n}\sum_{t=1}^{n} t$.

ANALYTICAL METHOD – LINEAR TREND

Estimation of Parameters: Least Squares Method

Alternatively, the slope b can be calculated as the ratio of the covariance between x_t and t to the variance of t:

$$\boxed{b = s_{xt}/s_t^2}$$

where:

$$s_{xt} = \frac{1}{n}\sum_{t=1}^{n}(t-\bar{t})(x_t-\bar{x}) \quad \text{(covariance between } t \text{ and } x_t\text{)}$$

$$s_t^2 = \frac{1}{n}\sum_{t=1}^{n}(t-\bar{t})^2 = \mathrm{Var}(t)$$

ANALYTICAL METHOD – LINEAR TREND

<u>**Estimation of Parameters: Least Squares Method**</u>

If $t = 1,2, \ldots, n$ — mean and variance of t

$$\bar{t} = \frac{1}{n}\sum_{t=1}^{n} t = \frac{n+1}{2}$$

$$s_t^2 = \mathrm{Var}(t) = \frac{1}{n}\sum_{t=1}^{n}(t-\bar{t})^2 = \frac{n^2-1}{12}$$

Residuals

$$e_t = x_t - (a + bt)$$

where e_t measures the deviation of the observed value from the estimated trend.

Note:
Other functional forms may be adopted for the trend (for example, quadratic, exponential, or logarithmic) if the data suggest a different pattern. This approach described above corresponds to the first analytical method (see Silvestre, 2007).

ANALYTICAL METHOD - LINEAR TREND: EXAMPLE

Observed Time Series Data (x_t):

t	x_t
1	5
2	7
3	9
4	11
5	13

Step 1 – Compute Means

$$\bar{t} = \frac{1+2+3+4+5}{5} = 3$$

$$\bar{x} = \frac{5+7+9+11+13}{5} = 9$$

ANALYTICAL METHOD - LINEAR TREND: EXAMPLE

Observed Time Series Data (x_t):

t	x_t
1	5
2	7
3	9
4	11
5	13

Step 2 – Compute Covariance and Variance

$$s_{xt} = \frac{1}{n}\sum_{t=1}^{5}(t-\bar{t})(x_t-\bar{x}) = \frac{1}{5}[(-2)(-4)+(-1)(-2)+(0)(0)+(1)(2)+(2)(4)]$$

$$= 4$$

$$s_t^2 = \frac{1}{5}\sum_{t=1}^{5}(t-\bar{t})^2 = \frac{1}{5}[(-2)^2+(-1)^2+0^2+1^2+2^2] = \frac{10}{5} = 2$$

ANALYTICAL METHOD - LINEAR TREND: EXAMPLE

Observed Time Series Data (x_t):

t	x_t
1	5
2	7
3	9
4	11
5	13

Step 3 – Compute Slope and Intercept

$$b = \frac{s_{xt}}{s_t^2} = \frac{4}{2} = 2$$

$$a = \bar{x} - b\bar{t} = 9 - 2 \cdot 3 = 3$$

ANALYTICAL METHOD - LINEAR TREND: EXAMPLE

Observed Time Series Data (x_t):

t	x_t
1	5
2	7
3	9
4	11
5	13

Step 4 – Trend Equation

$$\boxed{T_t = a + bt = 3 + 2t}$$

Step 5 – Residuals

$$\boxed{e_t = x_t - T_t}$$

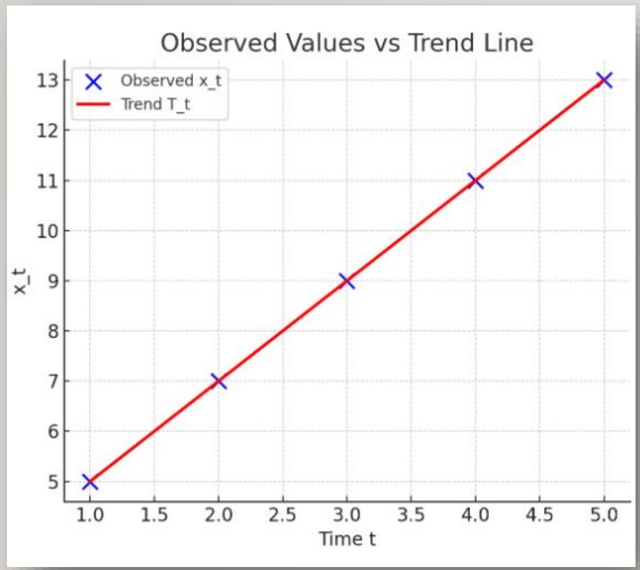
ANALYTICAL METHOD - LINEAR TREND: EXAMPLE

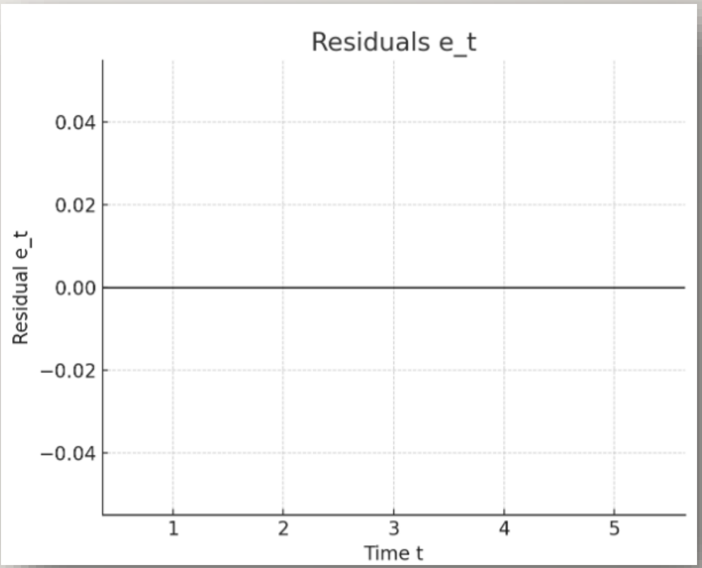
Observed Time Series Data (x_t):

t	$\mathbf{x_t}$	**Tt (Trend)**	**et (Residuals)**
1	5	5	0
2	7	7	0
3	9	9	0
4	11	11	0
5	13	13	0

All residuals are zero – the **linear trend perfectly fits** this simple data.

ANALYTICAL METHOD - LINEAR TREND: EXAMPLE





t	x_t	Tt	et
1	5	5	0
2	7	7	0
3	9	9	0
4	11	11	0
5	13	13	0

Note:
Residuals are rarely all zero in practice. In our case, this occurs because it is an academic example.

MOVING AVERAGE METHOD

Concept

- The **moving average (MA)** method smooths a time series by calculating averages over consecutive periods.
- The **period (n)** is the number of consecutive observations used to calculate each average.

Moving Average Formulas

a) Simple Moving Average (Period 3)

$$\boxed{\frac{x_1+x_2+x_3}{3}, \quad \frac{x_2+x_3+x_4}{3}, \quad \frac{x_3+x_4+x_5}{3}, \dots}$$

b) Simple Moving Average (Period 4)

$$\boxed{\frac{x_1+x_2+x_3+x_4}{4}, \quad \frac{x_2+x_3+x_4+x_5}{4}, \dots}$$

c) Centered Moving Average (Period 4)

$$\boxed{\frac{1}{2}\left(\frac{x_1+x_2+x_3+x_4}{4}+\frac{x_2+x_3+x_4+x_5}{4}\right)=\frac{1}{8}(x_1+2x_2+2x_3+2x_4+x_5), \dots}$$

PURPOSE OF MOVING AVERAGE METHOD

1. Reduce Random Fluctuations

- Original data can vary a lot from one period to another.
- The moving average **smooths these variations**, making the overall pattern easier to see.

2. Highlight Trends

- Helps to identify whether values are **increasing, decreasing, or stable** over time.

3. Facilitate Simple Forecasts

- The moving average can serve as a **basis to predict future values**, assuming the trend continues.

Note:
MA_3 denotes the moving averages with a period of 3.

Example:

- Original data: 10, 12, 15, 18, 20 → irregular fluctuations.
- MA_3: 12.33, 15.00, 17.67 → smoother curve, clearly showing the series is **increasing**.

CHOOSING THE MOVING AVERAGE PERIOD

What the 3-period and 4-period moving averages show:
They **smooth the time series**.
They **highlight the trend** (upward, downward, or stable).
MA_3 → reacts more quickly to short-term changes.
MA_4 → produces a smoother trend line.
They **do not allow forecasting**, as they rely only on past data.

Shorter periods (e.g., 3):

- Respond quickly to recent changes.
- Less smoothing, may still show fluctuations.

Longer periods (e.g., 4, 5, or more):

- Produce smoother curves.
- Better for identifying long-term trends, but slower to respond to recent changes.

Rule of thumb:

- The period should reflect the **seasonality or frequency** of the data.
- For **monthly data with seasonal cycles of 3 months**, use period 3.
- For **quarterly or yearly trends, use period 4, 5, or more** depending on the data.

MOVING AVERAGE METHOD: EXAMPLE

Observed Time Series Data (x_t):

t	x_t
1	10
2	12
3	15
4	18
5	20
6	23
7	25
8	28

a) Period 3 Moving Average

$$\frac{x_1+x_2+x_3}{3}=12.33,\quad \frac{x_2+x_3+x_4}{3}=15.00,\quad \frac{x_3+x_4+x_5}{3}=17.67,\ldots$$

b) Period 4 Moving Average

$$\frac{x_1+x_2+x_3+x_4}{4}=13.75,\quad \frac{x_2+x_3+x_4+x_5}{4}=16.25,\ldots$$

c) Centered Period 4 Moving Average

$$\frac{1}{2}\left(\frac{x_1+x_2+x_3+x_4}{4}+\frac{x_2+x_3+x_4+x_5}{4}\right)=15.00,\ldots$$

Difference between 4-period moving averages and centered moving averages

- The **4-period moving averages (MA_4)** are calculated by averaging **four consecutive observations**.
- The resulting value lies **between the 2nd and 3rd observations**, so it is **not centered** on a specific time period.
- To properly align the averages with the corresponding time periods, a **centered moving average** is obtained by averaging **two consecutive 4-period moving averages**.

In summary:

- **MA_4** → smooths the series but shifts the values.
- **Centered MA_4** → corrects the shift and aligns the trend with the central period.

MOVING AVERAGE (MA) METHOD: EXAMPLE

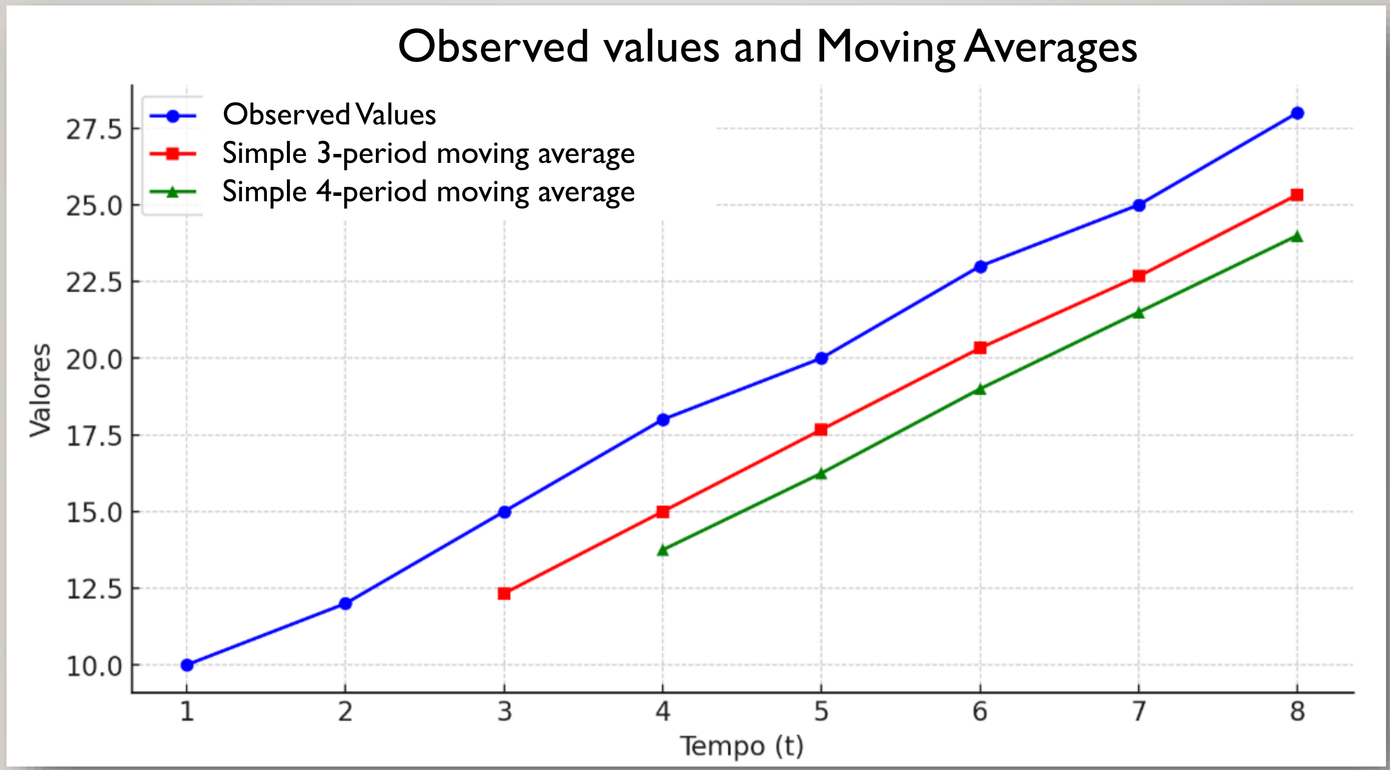
Observed Time Series Data (x_t):

t	x_t	MA - Period 3	MA - Period 4	Centered MA - Period 4
1	10	-	-	-
2	12	-	-	-
3	15	12.33	-	-
4	18	15.00	13.75	-
5	20	17.67	16.25	15.00
6	23	20.33	20.00	18.13
7	25	22.67	21.50	20.75
8	28	25.33	24.00	22.75

Note:

- **MA_3:** Simple 3-period moving average → smooths short-term fluctuations.
- **MA_4:** Simple 4-period moving average → smoother than **MA_3**.
- **Centered MA_4:** Average of two consecutive 4-period MAs → aligns with the center of the period.
- **Interpretation:** Shorter periods react faster to changes; longer or centered averages show long-term trends more clearly.

MOVING AVERAGE (MA) METHOD: EXAMPLE



LECTURE 10: SEASONALITY ANALYSIS OF A TIME SERIES

STUDYING SEASONAL MOVEMENTS

Seasonal movements repeat at regulat intervals.

There are **two types of seasonal patterns**:

Fixed Pattern

- The seasonal effect is **constant over time**.
- Example: Sales increase every December.
- **Note:** In this study, we will **only consider fixed patterns**.

Variable Pattern

- The seasonal effect **changes over time**, not perfectly repeated.
- Example: Tourist visits vary each year depending on weather events.

Key Difference:

- In the **additive model**, the seasonal component **does not depend on the trend**. It is **constant over time**, meaning that the seasonal effects have the same magnitude regardless of whether the series is at a low or high level
- In the **multiplicative model**, unlike the additive model, the seasonal component **depends on the level of the trend**. In simple terms, the higher the trend, the larger the seasonal effect.

Seasonality Models:

- **Additive:** Seasonal variation is **constant**, independent of the level of the series.

$$\boxed{Y_t = T_t + S_t + E_t}$$

- **Multiplicative:** Seasonal variation **changes proportionally** with the level of the series.

$$\boxed{Y_t = T_t \times S_t \times E_t}$$

STUDYING SEASONAL MOVEMENTS

Two methods to study seasonal movements:

- **Monthly averages (or medians) method**
- **Moving averages method**

Note:
The seasonal component is **not directly estimated**. Unlike the trend, which can be modeled (for example, by fitting a straight line or curve), the seasonal effect is **measured using seasonal indices**, which reflect the recurring pattern over time. These indices indicate how each period (month, quarter, etc.) deviates proportionally from the overall average.

MONTHLY AVERAGES METHOD

Steps (Silvestre's Method):

1. **Arrange the observations in a table**, reserving rows for months and columns for years (or vice versa).
2. **Calculate the sums** for each month (row) and for each year (column), and record them in the respective row and column.
3. **Compute the monthly averages** and the **overall average** (average of the monthly averages).
4. **Express each month's average as a proportion of the overall average** (for multiplicative seasonality).
   - For additive seasonality, consider the **difference from the overall average**.

Note:
This method (**Monthly Averages Method**) is **not used to estimate the seasonal component itself**, but rather to **identify and measure the seasonal pattern** through seasonal indices. The indices can then be used to **adjust the series** for seasonality in further analysis

MONTHLY AVERAGES METHOD - MULTIPLICATIVE SEASONALITY: EXAMPLE

Month	2021	2022	2023	**Row Sum**	**Row Mean**	**Seasonal Index**
Jan	120	130	125	375	125	0.82
Feb	140	135	145	420	140	0.91
Mar	150	155	160	465	155	1.01
Apr	160	165	170	495	165	1.08
May	180	175	185	540	180	1.18
Column Sum	750	760	785	2295	–	–
Column Mean	150	152	157	–	153	–

Explanations:

1. **Row Sum:** sum of values for each month across years.
2. **Row Mean:** average for each month across years.
3. **Seasonal Index (multiplicative):** row mean ÷ **overall mean.**
   - Overall mean = average of row means = (125+140+155+165+180)/5 = 153
4. **Column Sum:** sum of values for each year across months.
5. **Column Mean:** mean of each year across months.

Note: The detailed explanation of how to calculate the seasonal indices will be provided in the following slides.

Seasonal Index Interpretation:
Values **>1** → month above average activity
Values **<1** → month below average activity

MONTHLY AVERAGES METHOD MULTIPLICATIVE SEASONALITY: EXAMPLE

How to Calculate the Seasonal Index (Multiplicative Model)

1. **Calculate the Row Mean (monthly average)**
   - For each month, sum the values across the years and divide by the number of years.
   - **Example: January**

$$\text{Row Mean (Jan)} = \frac{120 + 130 + 125}{3} = 125$$

2. **Calculate the Overall Mean (global average)**
   - Average of all **row means**.
   - For 5 months:

$$\text{Overall Mean} = \frac{125 + 140 + 155 + 165 + 180}{5} = 153$$

MONTHLY AVERAGES METHOD MULTIPLICATIVE SEASONALITY: EXAMPLE

3. Calculate the Seasonal Index

- For each month:

$$\text{Seasonal Index} = \frac{\text{Row Mean}}{\text{Overall Mean}}$$

- **Example January:**

$$SI_{Jan} = \frac{125}{153} \approx 0.82$$

- **Example February:**

$$SI_{Feb} = \frac{140}{153} \approx 0.91$$

- **Example March:**

$$SI_{Mar} = \frac{155}{153} \approx 1.01$$

Note:
The detailed explanation of how to calculate the seasonal indices will be provided in the following slides.

MOVING AVERAGES METHOD FOR SEASONAL ANALYSIS

Definition:

- The **Moving Averages Method** is used to **smooth the time series** and isolate the seasonal component.
- Particularly useful for **series with a multiplicative or additive seasonal pattern**.

Objective:

To smooth the time series, eliminate short-term fluctuations, and identify the **seasonal component** by isolating the **trend** using **centered moving averages**.

Assumption:
At this stage, we assume that the time series is composed only of trend and seasonal components (T and S), that is, $Y_t = T_t + S_t$ (additive model) or $Y_t = T_t \times S_t$ (multiplicative model). The irregular component is not considered.

MOVING AVERAGES METHOD FOR SEASONAL ANALYSIS

Steps of the Method

1. Arrange the observations

Place the data in a table by **year (columns)** and **quarters (rows)**, e.g.

$X_{1I}, X_{1II}, X_{1III}, X_{1IV}, X_{2I}, X_{2II}, \ldots$

2. Compute the 4-quarter Moving Averages (MA(4))

Each moving average is obtained as:

$$MA(4)_t = \frac{X_{tI} + X_{tII} + X_{tIII} + X_{tIV}}{4}$$

Example:

$$MA(4)_1 = \frac{X_{1I} + X_{1II} + X_{1III} + X_{1IV}}{4}$$

$$MA(4)_2 = \frac{X_{2I} + X_{2II} + X_{2III} + X_{2IV}}{4}$$

Note: In this case, we are considering **quarterly data** (four quarters per year), so a 4-term moving average is used to smooth out seasonal effects and estimate the trend component.

MOVING AVERAGES METHOD FOR SEASONAL ANALYSIS

3. Calculate the Centered Moving Averages (CMA)

Since a 4-term moving average lies between two quarters, we must **center it** to align with the actual time period:

$$CMA_t = \frac{MA(4)_t + MA(4)_{t+1}}{2}$$

Or explicitly, using Silvestre's notation:

$$CMA = \frac{\left(\frac{X_{1I}+X_{1II}+X_{1III}+X_{1IV}}{4}\right) + \left(\frac{X_{2I}+X_{2II}+X_{2III}+X_{2IV}}{4}\right)}{2}$$

4. Compute the Seasonal Component

- **Multiplicative model:**

$$S_t = \frac{X_t}{CMA_t}$$

- **Additive model:**

$$S_t = X_t - CMA_t$$

These values represent the **seasonal effect** for each quarter.

MOVING AVERAGES METHOD FOR SEASONAL ANALYSIS

5. Average the Seasonal Effects by Quarter

Group all the seasonal effects of the same quarter (I, II, III, IV) and compute their mean to obtain the **Seasonal Indices**:

$$SI_I = \text{average of all } S_I \text{ values}$$

$$SI_{II} = \text{average of all } S_{II} \text{ values}$$

...and so on.

ESTIMATION OF SEASONAL INDICES USING THE MOVING AVERAGE METHOD: EXAMPLE

Assumption
At this stage, we assume that the time series contains only **trend (T)** and **seasonal (S)** components, i.e.

$$X_t = T_t \times S_t$$

This is a **multiplicative model**, and the **irregular component (I)** is ignored.
We also assume that the data are **quarterly**, so each year is divided into four quarters: $Q1, Q2, Q3, Q4$.

Year	Quarter	Observed Value (xt)
2023	Q1	120
2023	Q2	150
2023	Q3	180
2023	Q4	160
2024	Q1	130
2024	Q2	160
2024	Q3	200
2024	Q4	170

ESTIMATION OF SEASONAL INDICES USING THE MOVING AVERAGE METHOD: EXAMPLE

Step 1 – Arrange the observations

Arrange the observed values X_t in a chronological order, reserving one line for each period (e.g., quarter) and one column for each year.

Year	Q1	Q2	Q3	Q4
2023	120	150	180	160
2024	130	160	200	170

ESTIMATION OF SEASONAL INDICES USING THE MOVING AVERAGE METHOD: EXAMPLE

Step 2 – Compute the Moving Averages $MA(4)_t$

To smooth out the seasonal effect, compute 4-term moving averages:

$$MA(4)_t = \frac{X_t + X_{t+1} + X_{t+2} + X_{t+3}}{4}$$

Period	MA(4)t
Between Q1–Q4 2023	(120 + 150 + 180 + 160)/4 = 152.5
Between Q2–Q1 2024	(150 + 180 + 160 + 130)/4 = 155.0
Between Q3–Q2 2024	(180 + 160 + 130 + 160)/4 = 157.5
Between Q4–Q3 2024	(160 + 130 + 160 + 200)/4 = 162.5
Between Q1–Q4 2024	(130 + 160 + 200 + 170)/4 = 165.0

ESTIMATION OF SEASONAL INDICES USING THE MOVING AVERAGE METHOD: EXAMPLE

Step 3 – Center the Moving Averages

Because 4 is an even number of terms, the moving averages must be centered:

$$CMA_t = \frac{MA(4)_t + MA(4)_{t+1}}{2}$$

Centered at	CMAt
Q3 2023	(152.5 + 155.0)/2 = 153.75
Q4 2023	(155.0 + 157.5)/2 = 156.25
Q1 2024	(157.5 + 162.5)/2 = 160.00
Q2 2024	(162.5 + 165.0)/2 = 163.75

Note:
1. Window of 4 periods → no exact center.
2. Take the average of two consecutive MAs → centralizes on the middle period.
3. In your case, the CMA of Q1–Q4 2023 and Q2–Q1 2024 → centered on Q3-2023.

ESTIMATION OF SEASONAL INDICES USING THE MOVING AVERAGE METHOD: EXAMPLE

Step 4 – Compute the Ratios X_t / CMA_t

The ratio of each observation to its centered moving average provides the **seasonal factor** (and possibly a small irregular component):

Quarter	Xt	CMAt	Xt / CMAt
Q3 2023	180	153.75	1.17
Q4 2023	160	156.25	1.02
Q1 2024	130	160.00	0.81
Q2 2024	160	163.75	0.98

ESTIMATION OF SEASONAL INDICES USING THE MOVING AVERAGE METHOD: EXAMPLE

Step 5 – Average Ratios by Period

Group the ratios by quarter and calculate their mean — this gives the **seasonal index** for each quarter:

Quarter	Average Ratio (Seasonal Index)
Q1	0.81
Q2	0.98
Q3	1.17
Q4	1.02

Note:
In this example, there is only **one ratio available per quarter** (due to the limited number of years and centered moving averages). Therefore, the **seasonal index for each quarter is equal to that single ratio**.

- Q1 → 0.81
- Q2 → 0.98
- Q3 → 1.17
- Q4 → 1.02

These indices indicate how each quarter typically deviates from the trend: above (>1) or below (<1).

ESTIMATION OF SEASONAL INDICES USING THE MOVING AVERAGE METHOD: EXAMPLE

Step 6 – Normalize the Indices

Normalize so that their average equals 1 (for the multiplicative model):

$$S_i^* = \frac{S_i}{\text{Mean of all indices}}$$

Mean of all indices = (0.81 + 0.98 + 1.17 + 1.02)/4 = 0.995

Quarter	Normalized Si*
Q1	0.81
Q2	0.99
Q3	1.18
Q4	1.03

ESTIMATION OF SEASONAL INDICES USING THE MOVING AVERAGE METHOD: EXAMPLE

Step 7 – Interpretation

- **Q3 = 1.18** → On average, values in the 3rd quarter are **18% higher** than the yearly average.
- **Q1 = 0.81** → On average, values in the 1st quarter are **19% lower** than the average.

Notes:

- In this example, quarterly data for two consecutive years (2023–2024) were used.
- It is assumed that the series contains only **trend (T)** and **seasonal (S)** components:
- $X_t = T_t \times S_t$
- The irregular component is **not considered** at this stage.

THANKS!

Questions?