



Normal Season - Second Midterm Test (MT2) - December 15, 2023 - Time: 1 hour

Justify your answers carefully

The values in [*.*] represent the rating of the corresponding question on a 0-10 scale

1. [2.0] Considering δ_a the Dirac measure at $a \in \mathbb{R}$ and m_A the Lebesgue measure on $A \in \mathcal{B}(\mathbb{R})$, determine the distribution function of the probability measure on $\mathbb{R}$ defined by

$$\mu(B) = \frac{1}{4}\delta_{-1}(B) + \frac{1}{2}m_{[0,1]}(B) + \frac{1}{4}\delta_2(B), \quad \text{for all } B \in \mathcal{B}(\mathbb{R}).$$

2. [3.0] Let X be a continuous random variable with Exponential distribution with parameter $\lambda > 0$, whose probability density function is

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}.$$

- (a) Compute the expected value of X .
- (b) Determine the distribution function of X .

3. [3.0] Let Ω be a set, $\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2$ be σ -algebras of Ω , and $X, Y : \Omega \rightarrow \mathbb{R}$ two random variables defined on Ω . Prove that:

- (a) If $\mathcal{G} \subset \mathcal{F}_1$ and $\mathcal{F}_1, \mathcal{F}_2$ are independent σ -algebras, then $\mathcal{G}$ and $\mathcal{F}_2$ are also independent.
- (b) X and Y are independent if and only if $\sigma(X)$ and $\sigma(Y)$ are independent.

4. [2.0] Let $\Omega = [0, 1]$ with the σ -algebra $\mathcal{B}(\Omega)$, and m the Lebesgue measure on Ω . Considering $X, Y : \Omega \rightarrow \mathbb{R}$ two random variables defined as

$$X(x) = x \quad \text{and} \quad Y(x) = \begin{cases} 1, & \text{if } 0 \leq x < \frac{1}{2} \\ 2, & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases},$$

explicitly determine the random variable $\mathbb{E}(X | Y)$.