Prob. Theory and Stochastic Processes — 08/01/2026. (1)

— Resit Exam.

Part I

a) $\{\emptyset, \{i, s\}, \{e, g\}, \{i, s, e, g\}\}$

b) infinite countable

- Let n number of A in O.
- $f(n) = n+3$.

c) $\mu(A \cup B) = 13/20$

A and B are dependent

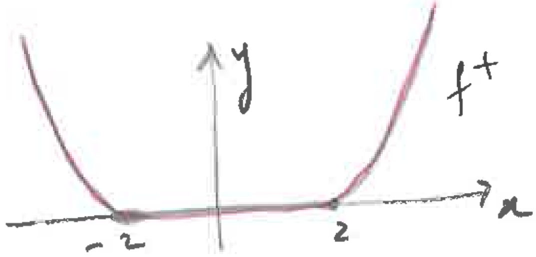
d) $m(A) = 4$

- $$\sum_{n=1}^{+\infty} \frac{\delta_{1/n}(B)}{2^n} = 1$$

- $\mathcal{C} = \{1, 2\}$ for instance

$(m \times \delta_\pi)(A \times A) = 4$

e)



f) continuous, open

g) $[0,1]$, Riemann

h) pointwisely

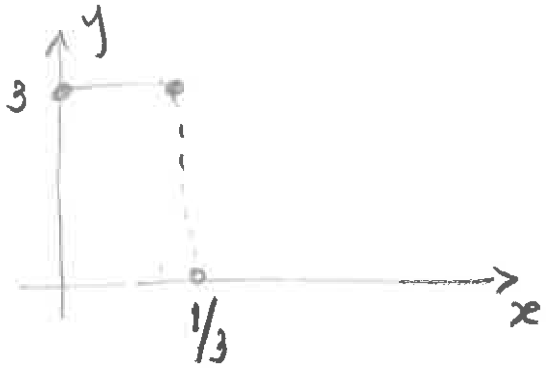
$$\lim_{n\to+\infty} \sup_{x\in[0,1]} |f_n(x) - f(x)| \neq 0$$

For $x \in [0,1[$, $|f_n(x) - f(x)| = |x^{2n}|$

when $x \to 1$, $|x^{2n}| \to 1$.

i) Cantor, 0, infinite uncountable.

j)



$$0 = \int_{[0,+\infty[} \lim \varphi_n \, dx < \lim_{n\to+\infty} \int \varphi_n \, dm = 1.$$

φ_n is not bounded by an integrable map

k) $\int_{\mathbb{R}} \sqrt{1+x^2}\, d\mu(x) = \int_{[0,1]} x\sqrt{1+x^2}\, dm(x)$

$= \int_0^1 x\sqrt{1+x^2}\, dx = \dfrac{2\sqrt{2}-1}{3}$.

l) $1/2$

m) . $m([0,\frac{1}{3}]) = 2/3$

. $\delta_0 \perp m$ because $\delta_0(\{0\}) = 1$ and $m(\{0\}^c) = 1$

. Radon-Nikodym

. $f(0) = 2$.

n. (1) $\alpha = p\,\delta_{c_1} + (1-p)\,\delta_{c_2}$

Bernoulli

$E(v) = \int x\, d\alpha = p\,c_1 + (1-p)\,c_2$

(o) recurrent positive

. $(1/4, \frac{1}{2}, \frac{1}{4})$

. aperiodic, irreducible

. 4, 2 and 4.

Part II

1) $$E(|X|^k) = \int_{\mathbb{R}} |X|^k \, dP \geq$$

$$\geq \int_A |X|^k \, dP \geq \int_A \lambda^k \, dP = \lambda^k P(A)$$

$$\underset{\substack{\uparrow \\ |X| \geq \lambda}}{P(A)} \leq \frac{E(|X|^k)}{\lambda^k}$$

2) a)

$$f_n(x) \xrightarrow{n} f(x) = \begin{cases} x & 0 \leq x < 1 \\ 1/4 & x = 1 \\ 0 & x > 1 \end{cases}$$

b) $0 \leq x < 1$

$$|f_n(x)| \leq \frac{x}{(x^{2n}+1)^2} \underset{\substack{\uparrow \\ x^{2n} \geq 0}}{\leq} x \underset{\substack{\uparrow \\ x \in [0,1[}}{\leq} 1$$

$x \geqslant 1$

$$|f_n(x)| \leq \frac{x}{(x^{2n}+1)^2} \leq \frac{x}{(x^2)^2} = \frac{1}{x^3} = x^{-3}$$

$\forall n \in \mathbb{N},\ x^{2n}+1 \geqslant x^{2n} \geqslant x^2$

c) • $|f_n(x)| \leq g(x)$, $\forall x \in \mathbb{R}_0^+$, $\forall n \in \mathbb{N}$

• g is integrable.

$$\int_0^1 g(x)\,dx = 1$$

$$\int_1^{+\infty} g(x)\,dx = \left[\frac{x^{-2}}{-2}\right]_1^{+\infty} = \left[-\frac{1}{2x^2}\right]_1^{+\infty} = 0 + \frac{1}{2}$$

$$\Rightarrow \boxed{\int_0^{+\infty} g(x)\,dx = \text{[illegible]} < +\infty}$$

Therefore, by the dominated Convergence theorem, we have

$$\lim_{n\to+\infty} \int_{[0,+\infty[} f_n(x)\,dm(x) = \int_0^1 x\,dx + \int_1^1 \frac{1}{4}\,dx + \int_1^{+\infty} 0\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^1 + 0 + 0 = \frac{1}{2}$$

3. Let us prove by "Counterparts"

$$(A \Longrightarrow B) \Longrightarrow (\sim B \Longrightarrow \sim A)$$

- $X-Y$ and $X+Y$ are independent

$$\Downarrow \text{ result of the course}$$

$$E((X-Y)(X+Y)) = E(X-Y) \cdot E(X+Y)$$

$$\Downarrow$$

$$E(X^2-Y^2) = (E(X)-E(Y))(E(X)+E(Y))$$

$$\Downarrow$$

$$E(X^2)-E(Y^2) = E(X)^2 - E(Y)^2$$

$$\Downarrow$$

$$E(X^2)-E(X)^2 = E(Y^2)-E(Y)^2$$

$$\Downarrow$$

$$Var(X) = Var(Y)$$

4)

$E(Z|W)(w)$ for $w \in \{x \in \mathbb{R} \; W(x) = 0\}$

$$E(Z|W)(0) = \frac{1}{P(W=0)} \cdot \int_{W=0} Z \, dP = 4 \cdot \frac{1}{5}$$

$$P(W=0) = P(X \in [0,1]) =$$

$$= F_X(1) - F_X(0) = \frac{1}{4}$$

$$\int_0^1 \sqrt{X} \cdot \frac{2X}{4} \, dx = \int_0^1 \frac{1}{2} x^{3/2} \, dx = \left. \frac{x^{5/2}}{5/2 \cdot 2} \right|_0^1$$

$$= \frac{1}{5}$$

5

a) $\phi(t) \rightarrow$ Taylor expansion of the characteristic map of X_n at $t=0$ (does not depend on $\underline{n}$)

$$\phi'(0) = i\, m_1 = i\, E(X) = 0.$$

$$\phi''(0) = i^2 m_2 = i\, E(X^2) = i^2 \sigma^2 = -\sigma^2$$

$$Var(X_n) = E(X^2) - \underbrace{E(X)^2}_{=0} = E(X^2)$$

$$\phi(t) = 1 + \phi'(0)\, t + \frac{\phi''(0)\, t^2}{2} + \cdots$$

$$= 1 + 0 - \frac{\sigma^2}{2} t^2 + \underbrace{\cdots}_{\text{monomials of order } \geq 3 \text{ in } t}$$

$$\phi\left(\frac{t}{\sqrt{n}}\right) = 1 - \frac{\sigma^2}{2} \frac{t^2}{n} + \cdots$$

b) $$\phi\left(\frac{t}{\sqrt{n}}\right)^n = \left(1 - \frac{\sigma^2}{2}\frac{t^2}{n} + \cdots\right)^n$$

$$\lim_{n\to+\infty} \phi\left(\frac{t}{\sqrt{n}}\right)^n = \lim_{n\to+\infty}\left(1 - \frac{\sigma^2}{2}\frac{t^2}{n} + \cdots\right)^n$$

$$= e^{-\frac{\sigma^2 t^2}{2}} = E\left(e^{itY}\right)$$

$$Y = \frac{X_1 + \cdots + X_n}{\sqrt{n}}$$

$\phi\left(\frac{t}{\sqrt{n}}\right)$ is the characteristic map of $\frac{X_j}{\sqrt{n}}$, $j=1,\dots,n$

$\phi\left(\frac{t}{\sqrt{n}}\right)^n$ is the characteristic map of $\dfrac{\sum_{j=1}^{n} X_j}{\sqrt{n}}$

$S_n = \sum_{j=1}^{n} X_j$, then $\frac{S_n}{\sqrt{n}} \xrightarrow{n} Y$

$Y \sim N(0, \sigma^2)$.

(This is a particular case of the Central Limit theorem)