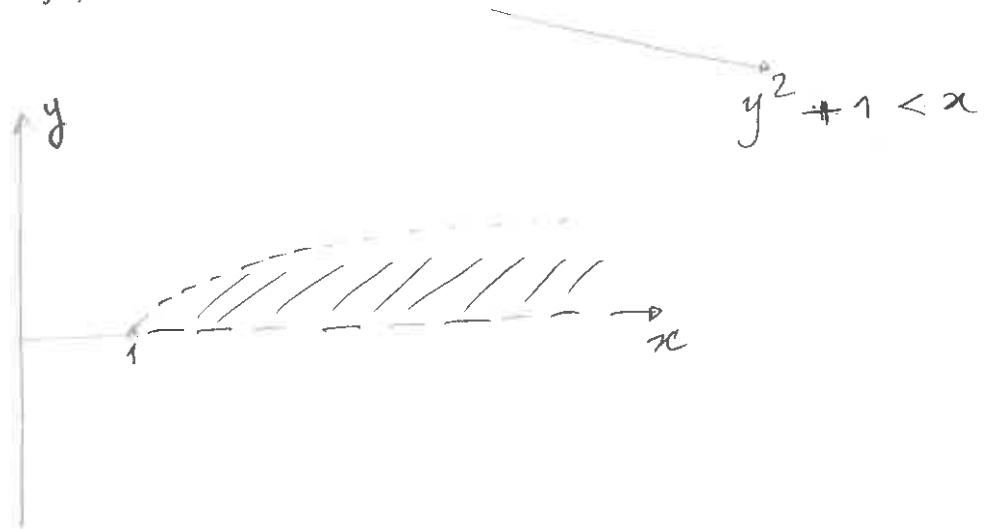


Part I.

a) $D_f = \{(x, y) \in \mathbb{R}^2 : x - y^2 - 1 > 0 \wedge y > 0\}$



- b) $(0,0)$ is not an interior / accumulation point
• $\text{int } \Omega = \{(x, y) \in \mathbb{R}^2 : |x-5| < 1 \wedge 4 < y < 7\}$
• Ω is not open.

c) $\lim f = [-4, 2]$

• $f \circ g$ has two zeros: e and e^{-1}

• $\lim_{n \rightarrow +\infty} f\left(-\frac{1}{n}\right) = 2$

d) $f(x, y) = x^2 y^3 - \sin y$

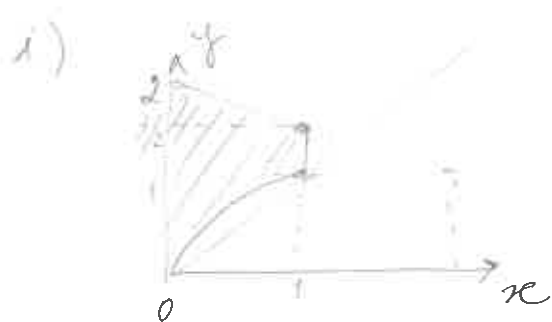
e) $H_f(3, 2) = \begin{pmatrix} -3 & 0 \\ 0 & -5 \end{pmatrix}$ (for instance)

f) $\frac{\partial^2 f}{\partial x^{10} \partial y^2}(x, y) = 3^8 \cos(3x)$

g) $M = \{(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\} : 1 \leq x^2 + y^2 \leq 2\}$ (for instance)

Weierstrass

h) I_2 is not convex.



$a \in [1, 3/2]$

H_a is upper semi-continuous

$\text{Fix } H_1 = \{0, 1\}$

j) Banach fixed point theorem.

$$\lim_{n \rightarrow \infty} f^n(0,0) = \underbrace{(2,2)}_{\text{fixed point of } f}.$$

$$k) \begin{cases} 6y - 2 \cdot 8\mu = 0 \\ 6x - 3\mu = 0 \\ \mu \leq 0 \\ \mu(8x + 3y - 17) = 0 \end{cases}$$

$$\boxed{\mu = 2}$$

l) separable

$$\frac{y'}{y^2} = -\frac{x}{x^2+1} \Leftrightarrow -\frac{1}{y} = -\frac{1}{2} \ln(x^2+1) + C \quad C \in \mathbb{R}$$

$$\Leftrightarrow y = \frac{1}{\frac{1}{2} \ln(x^2+1) + C}, \quad C \in \mathbb{R}$$

$$\text{Since } y(0)=2 \Leftrightarrow 2 = \frac{1}{\underbrace{\frac{1}{2} \ln 1}_{=0} + C} \Leftrightarrow C = \frac{1}{2}$$

$$\boxed{y(x) = \frac{2}{\ln(x^2+1) + 1}}$$

$$D = \mathbb{R}$$

$$(m) \quad \begin{cases} y' = \sqrt{y-1} \\ y(0) = 1 \end{cases}$$

$$f(y) = \sqrt{y-1}, \quad D_f = [1, +\infty[$$

$$f'(y) = \frac{1}{2\sqrt{y-1}}$$

$$\lim_{y \rightarrow 1^+} f'(y) = +\infty$$

f is not Lipschitz at $y = 1$ ($x = 0$).

$$(n) \quad (0,0) \text{ and } (1,1)$$

$$f(x,y) = (-x+xy, y-xy)$$

$$Df(x,y) = \begin{pmatrix} -1+y & x \\ -y & 1-x \end{pmatrix}$$

$$Df(1,1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x \end{cases}$$

$(1,1)$ is non-hyperbolic because the eigenvalues of $Df(1,1)$ are $\pm i$ (they have zero real part).

(5)

$$(*) \quad p' = 20p - 10p^2 \quad \Leftrightarrow \quad p' = 10p(2-p)$$

equilibria $p=0$ and $p=2$



- monotonic increasing
- $\lim_{t \rightarrow \infty} p(t) = 2$
- inflexion point where $p=1$

(p) Hamiltonian:

$$H(t, x, u, p) = 1 - t x - u^2 + p \cdot x$$

quality.

$$\frac{\partial H}{\partial u} = 0 \Leftrightarrow -2u + p = 0 \Leftrightarrow u = p/2$$

hamiltonian equation:

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial p} \\ \dot{p} = -\frac{\partial H}{\partial x} \end{cases} \Leftrightarrow \begin{cases} \dot{x} = u \\ \dot{p} = +t \end{cases} \Leftrightarrow \begin{cases} \dot{x} = p/2 \\ \dot{p} = t \end{cases}$$

Transversality condition

$$\begin{aligned} [p \ 1] = 0 &\Leftrightarrow \frac{t^2}{2} + C_1 = 0 \Leftrightarrow C_1 = -1/2 \\ &\Leftrightarrow \frac{t^2}{2} - \frac{1}{2} = 0 \end{aligned}$$

(16)

$$p(t) = \frac{t^2}{2} - \frac{1}{2}$$

$$\dot{x}(t) = \frac{t^2}{4} - \frac{1}{4} \Rightarrow x(t) = \frac{t^3}{12} - \frac{1}{4}t + C_2$$

$$\begin{matrix} x(0)=1 \\ \Rightarrow \end{matrix} C_2 = 1$$

$$\therefore x^*(t) = \frac{t^3}{12} - \frac{1}{4}t + 1 \quad u^*(t) = \frac{t^2}{4} - \frac{1}{4}$$

H is concave in (x, u)

$$\nabla H(x, u) = (-t; -2u + p)$$

$$H_{xx}(x, u) = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} \quad \text{eigenvalues} \leq 0$$

Part II

1 a) $*[0,1]^2$ is a compact and convex
(it is a square with boundary).

* F is continuous (the components are polynomials).

$$* F([0,1]^2) \subset [0,1]^2$$

(7)

Proof:

$$(x,y) \in [0,1]^2 \Rightarrow 0 \leq x+y \leq 2$$

$$\Rightarrow 0 \leq \frac{(x+y)^2}{4} \leq 1$$

$$\Rightarrow \frac{(x+y)^2}{4} \in [0,1]. \quad \checkmark$$

$$(x,y) \in [0,1]^2 \Rightarrow x \in [0,1] \quad \checkmark$$

$$b). \quad F(x,y) = (x,y) \Leftrightarrow \begin{cases} \frac{(x+y)^2}{4} = x \\ x = y \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x^2 = x \\ \text{---} \end{cases} \Leftrightarrow \begin{cases} x=0 \\ y=0 \end{cases} \quad \checkmark \quad \begin{cases} x=1 \\ y=1 \end{cases}$$

Fixed points of F : $(0,0)$ and $(1,1)$.

2) Lagrangian:

$$L(x, y, z, \lambda) = \frac{x}{2} + \frac{y}{2} - 2z - \lambda \left(\frac{x^2}{2} + \frac{y^2}{4} + z^2 - 1 \right)$$

$$\partial L(x, y, z, \lambda) = \vec{0} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \frac{1}{2} - \lambda x = 0 \\ \frac{1}{2} - \lambda \frac{y}{2} = 0 \\ -2 - 2\lambda z = 0 \\ \frac{x^2}{2} + \frac{y^2}{4} + z^2 - 1 = 0 \end{cases} \Leftrightarrow \begin{cases} x = \frac{1}{2\lambda} \\ y = \frac{1}{\lambda} \\ z = -\frac{1}{\lambda} \\ \frac{\left(\frac{1}{2\lambda}\right)^2}{2} + \frac{\left(\frac{1}{\lambda}\right)^2}{4} + \frac{\left(-\frac{1}{\lambda}\right)^2}{1} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \text{---} \\ \text{---} \\ \text{---} \\ \frac{1}{8\lambda^2} + \frac{1}{4\lambda^2} + \frac{1}{\lambda^2} = 1 \end{cases} \Leftrightarrow \begin{cases} \text{---} \\ \text{---} \\ \text{---} \\ \frac{11}{8\lambda^2} = 1 \end{cases}$$

$$\Rightarrow \begin{cases} \text{---} \\ \text{---} \\ \text{---} \\ 8\lambda^2 = 11 \end{cases} \Rightarrow \begin{cases} x = \pm \frac{\sqrt{2}}{\sqrt{11}} \\ y = \pm 2 \frac{\sqrt{2}}{\sqrt{11}} \\ z = \mp \frac{2\sqrt{2}}{\sqrt{11}} \\ \lambda = \pm \frac{\sqrt{11}}{2\sqrt{2}} \end{cases}$$

Candidates:

⑨

$$A \rightarrow \left(\frac{\sqrt{2}}{\sqrt{11}}, \frac{2\sqrt{2}}{\sqrt{11}}, -\frac{2\sqrt{2}}{\sqrt{11}} \right)$$

$$B \rightarrow \left(-\frac{\sqrt{2}}{\sqrt{11}}, -\frac{2\sqrt{2}}{\sqrt{11}}, \frac{2\sqrt{2}}{\sqrt{11}} \right).$$

$$f(A) = \frac{\sqrt{2}}{2\sqrt{11}} + \frac{\sqrt{2}}{\sqrt{11}} + \frac{4\sqrt{2}}{\sqrt{11}} =$$

$$= \frac{\sqrt{2} + 2\sqrt{2} + 8\sqrt{2}}{\sqrt{11}} = \frac{11\sqrt{2}}{\sqrt{11}} = \sqrt{22} > 0$$

$$f(B) = -\sqrt{22} < 0.$$

global maximum is $\sqrt{22}$.

③

$$y_p(x) = 2x - 1$$

$$y_p'(x) = 2$$

$$y_p''(x) = 0$$

$$\Rightarrow \begin{aligned} y'' + \alpha y' + 6y &= 12x + 4 \\ 0 + 2\alpha + 6(2x - 1) &= 12x + 4 \end{aligned}$$

$$\Leftrightarrow 2\alpha - 6 = 4$$

$$\Leftrightarrow \boxed{\alpha = 5}.$$

$$y'' + 5y' + 6y = 12x + 4$$

$$P(\lambda) = \lambda^2 + 5\lambda + 6$$

$$P(\lambda) = 0 \Leftrightarrow \lambda = -2 \vee \lambda = -3$$

Solution

$$y(x) = C_1 e^{-2x} + C_2 e^{-3x} + 2x - 1, x \in \mathbb{R}.$$

We know that

$$\begin{cases} y(0) = -1 \\ y'(0) = 1 \end{cases} \Leftrightarrow \begin{cases} C_1 + C_2 - 1 = -1 \\ -2C_1 - 3C_2 + 2 = 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} C_1 + C_2 = 0 \\ -2C_1 - 3C_2 = -1 \end{cases} \Leftrightarrow \begin{cases} C_1 = -1 \\ C_2 = 1 \end{cases}$$

$$\therefore y(x) = e^{-2x} + e^{-3x} + 2x - 1, x \in \mathbb{R}$$

$$4) \begin{cases} x' = x + 2y \\ y' = -2x + y \end{cases}$$

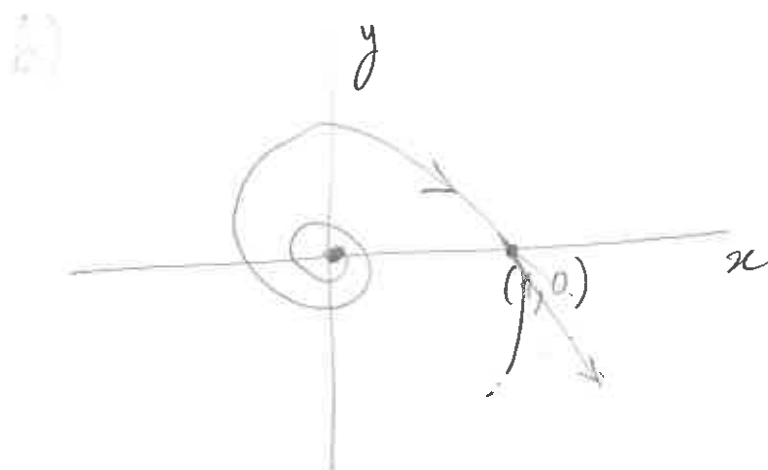
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$$a) \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}}_A \begin{pmatrix} x \\ y \end{pmatrix}$$

$$p(\lambda) = \det \begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 + 4$$

$$p(\lambda) = 0 \implies 1-\lambda = \pm 2i \implies \boxed{\lambda = 1 \pm 2i}$$

Since $\operatorname{Re}(\lambda) = 1 > 0$, then $(0,0)$ is unstable.



$$\begin{aligned} x' &= 1 \\ y' &= -2 \end{aligned}$$

5.

12

Euler-Lagrange equation: $F(t, x, \dot{x}) = 2t x + \dot{x}^2$

$$\frac{\partial F}{\partial x} - \frac{d}{dt} \frac{\partial F}{\partial \dot{x}} = 0$$

$$\Leftrightarrow 2t - \frac{d}{dt} (2\dot{x}) = 0 \Leftrightarrow$$

$$\Leftrightarrow 2t - 2\ddot{x} = 0 \Leftrightarrow \ddot{x} = t \Leftrightarrow \dot{x}(t) = \frac{t^2}{2} + C_1$$

$$\Leftrightarrow x(t) = \frac{t^3}{6} + C_1 t + C_2$$

Since:

$$\begin{cases} x(0) = 0 \\ x(1) = 7/6 \end{cases} \Leftrightarrow \begin{cases} C_2 = 0 \\ \frac{1}{6} + C_1 = \frac{7}{6} \end{cases} \Leftrightarrow \begin{cases} C_2 = 0 \\ C_1 = 1 \end{cases}$$

$$x(t) = \frac{t^3}{6} + t, \quad t \in [0, 1]$$

$$\nabla F(x, \dot{x}) = (2t, 2\dot{x})$$

$$H F(x, \dot{x}) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{eigenvalues } \geq 0$$

F is convex in $(x, \dot{x})$: $x^*(t) = \frac{t^3}{6} + t, \quad t \in [0, 1]$
is the solution.