Mathematical Programming I

BSc in Applied Mathematics for Economics and Management (MAEG)

ISEG
Lisbon School
of Economics
& Management
Universidade de Lisboa

2025-2026

Introduction to Linear Programming

Linear Programming

▶ **Programming** $\rightarrow$ Planning

▶ **Linear** $\rightarrow$ all involved functions are linear functions

Problem:
▶ Model (formalize)
▶ Resolution
  ▶ Graphical solution
  ▶ Apply the algorithm
  ▶ Use software
▶ Interpretation of results

Model or Formulation

Model or Formulation

Model or Formulation

A model or formulation is a schematic description of a system that takes into account its known properties and can be used to know it better, study its characteristics, and to better program the decisions.

To write a mathematical programming model we must answer the following questions:

- ▶ Which are the decisions to take?
- ▶ Under which constraints is the decision taken?
- ▶ Which is the appropriate criteria that allow an evaluation of the different alternatives of each decision?

Mathematical Programming Formulation

Mathematical Programming Formulation
(Operations Research Modelling)

1. **Define the decision variables** (one for each decision to be made)

2. **Define the objective and the objective function**

3. **Specify the functional constraints** (due to limited resources, minimum or maximum requirements, relationships between decisions, or similar considerations)

4. **Specify the variables' domain constraints** (non-negativity or sign constraints)

Model or Formulation

A typical mathematical programming model (or formulation) is organized as follows:

General Structure

minimize or maximize   an objective function

subject to   a set of constraints

Both the objective function and the constraints are mathematical functions defined in terms of the decision variables.

Model or Formulation

Let $x \in \mathbb{R}^n$ denote the vector of decision variables.
A general mathematical programming model can be written as:

$$\text{minimize (or maximize)} \quad z = f(x)$$

$$\begin{array}{lll}
\text{subject to} & g_i(x) \leq b_i, & i \in I_1 \quad (1) \\
& g_i(x) \geq b_i, & i \in I_2 \quad (2) \\
& g_i(x) = b_i, & i \in I_3 \quad (3) \\
& & \\
& x_j \geq 0, & j \in J_1 \quad (4) \\
& x_j \leq 0, & j \in J_2 \quad (5) \\
& x_j \in \mathbb{R}, & j \in J_3 \quad (6)
\end{array}$$

- x_j are the decision variables, with $x = (x_j) \in \mathbb{R}^n$, where $n = |J| = |J_1 \cup J_2 \cup J_3|$.

- f and $g_i, i \in I$, are real functions from $\mathbb{R}^n \to \mathbb{R}$.

- $z = f(x)$ is the objective function to be optimized.

- Constraints (1)–(3) are the functional constraints, with $m = |I| = |I_1 \cup I_2 \cup I_3|$.

- Constraints (4)–(6) are the domain (or sign) constraints of the variables.

Model or Formulation

Some types of mathematical programming models

in case of

▶ a Linear Programming (LP) model, all involved functions are linear functions

▶ an Integer Linear Programming (ILP) model, all involved decision variables are integer variables

▶ a Non-Linear Programming (NLP) model, some involved functions are nonlinear functions

Model or Formulation

General Form of a Linear Programming Problem

$$\text{minimize or maximize} \quad z = \sum_{j \in J} c_j x_j$$

$$\text{subject to} \quad \sum_{j \in J} a_{ij} x_j \leq b_i, \quad i \in I_1 \quad (1)$$

$$\sum_{j \in J} a_{ij} x_j \geq b_i, \quad i \in I_2 \quad (2)$$

$$\sum_{j \in J} a_{ij} x_j = b_i, \quad i \in I_3 \quad (3)$$

$$x_j \geq 0, \qquad j \in J_1 \quad (4)$$
$$x_j \leq 0, \qquad j \in J_2 \quad (5)$$
$$x_j \in \mathbb{R}, \qquad j \in J_3 \quad (6)$$

Parameters

- c_j: cost coefficient of decision variable x_j in the objective function.
- a_{ij}: technological coefficient of decision variable x_j in constraint i.
- b_i: right-hand side (RHS) of functional constraint i.

Model or Formulation

Don't forget! To write a model we must answer the following questions:

▶ Which are the decisions to take?

define the decision variables

▶ Under which constraints is the decision taken?

define the constraints

▶ Which is the appropriate criteria that allow an evaluation of the different alternatives of each decision?

define the objective function

Example of an LP model with two variables

The Reddy Mikks Company (RM) produces both interior and exterior paints from two raw materials, M1 and M2. The necessary quantity of materials (tons of raw material per ton of paint) are displayed in the following table

Product	Interior Paint	Exterior Paint	max daily availability (tons)
Raw Material M1 (tons)	4	6	24
Raw Material M2 (tons)	2	1	6
Profit ($\times \$1000$) / ton	4	5	

The daily demand for interior paint shall not exceed that of exterior paint by more than 1 ton.
The maximum daily demand for interior paint is 2 tons.
RM wants to determine the mix to produce that will maximise the total daily profit.

Example of an LP model with two variables

x_1 tons of interior paint produced daily

x_2 tons of exterior paint produced daily

$$\max \quad z = 4x_1 + 5x_2, \ \times \$1000 \ \leftarrow \text{total daily profit}$$

s. to: $4x_1 + 6x_2 \leq 24$, $\leftarrow$ raw material M1

$\quad\quad\ 2x_1 + x_2 \leq 6$, $\quad \leftarrow$ raw material M2

$\quad\quad\ x_1 - x_2 \leq 1$, $\quad\ \leftarrow$ interior paint $\leq$ exterior paint $+1$ ton

$\quad\quad\ x_1 \quad\ \leq 2$, $\quad\ \leftarrow$ max daily demand of interior paint is 2 tons

$\quad\quad\ x_1, x_2 \geq 0$, $\quad\ \leftarrow$ no negative production

Solving with the graphical method

Graphical method

1. Graphically represent the set of feasible solutions, identifying all the constraints and the half-space defined by each constraint. The feasible region is the intersection of the half-space defined by all the constraints (functional and sign).

2. If the feasible region is empty, the problem is infeasible. STOP.

3. Graph the gradient of the objective function and the line corresponding to the objective function at any arbitrary point. Set $z = a$ (a fixed and arbitrarily selected) and find the best value a such that the line $z = a$ intersects the feasible region.

   3.1 If one exists, identify the optimal solution, or set of optimal solutions, from among all the feasible solutions.

   3.2 Otherwise, conclude the problem is unbounded (there is no optimal solution).

Example of the graphical method: the RM model

Use the graphical method to find the optimal solutions of the Reddy Mikks LP.

$$\max \quad z = 4x_1 + 5x_2,$$

$$
\begin{aligned}
\text{s. to:} \quad & 4x_1 + 6x_2 \leq 24, \\
& 2x_1 + x_2 \leq 6, \\
& x_1 - x_2 \leq 1, \\
& x_1 \quad\quad \leq 2, \\
& x_1, x_2 \geq 0
\end{aligned}
$$

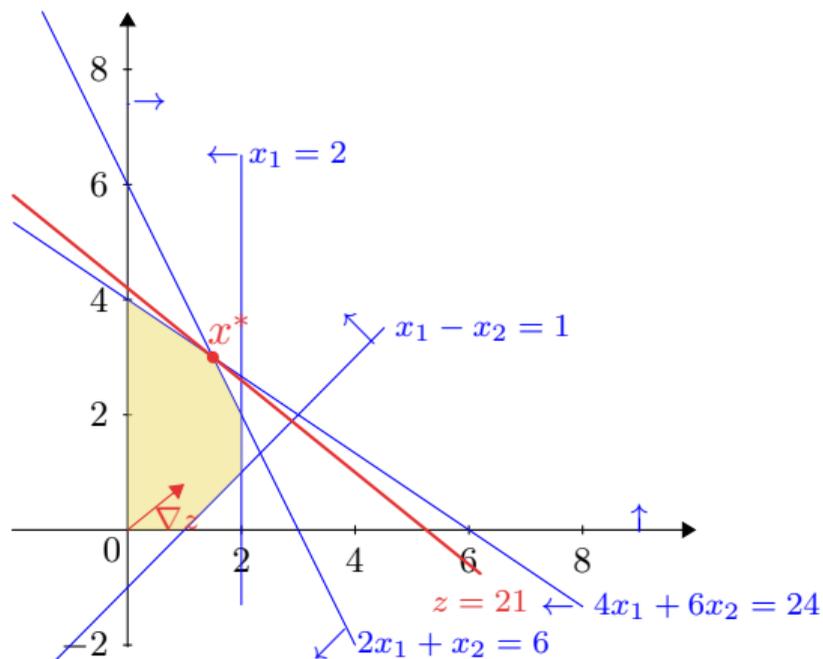# Example of the graphical method: the RM model

Step 1

Step 2

Step 3
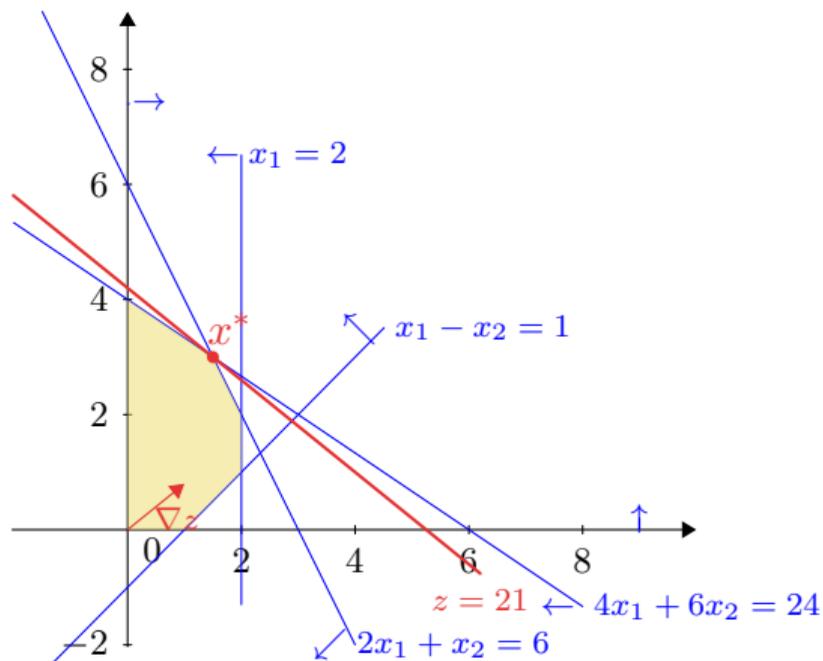
Example of the graphical method: the RM model

$\nabla z = (4, 5)$
the optimal solution is $x^* = (\frac{3}{2}, 3) = (1.5, 3)$ with value $z^* = 21$

Example 1

Use the graphical method to identify the optimal solutions of the following LP.

$$\max \quad z = 5x_1 + 4x_2,$$

$$
\begin{aligned}
\text{s. to:} \quad & 6x_1 + 4x_2 \leq 24, \\
& x_1 + 2x_2 \leq 6, \\
& -x_1 + x_2 \leq 1, \\
& x_2 \leq 2, \\
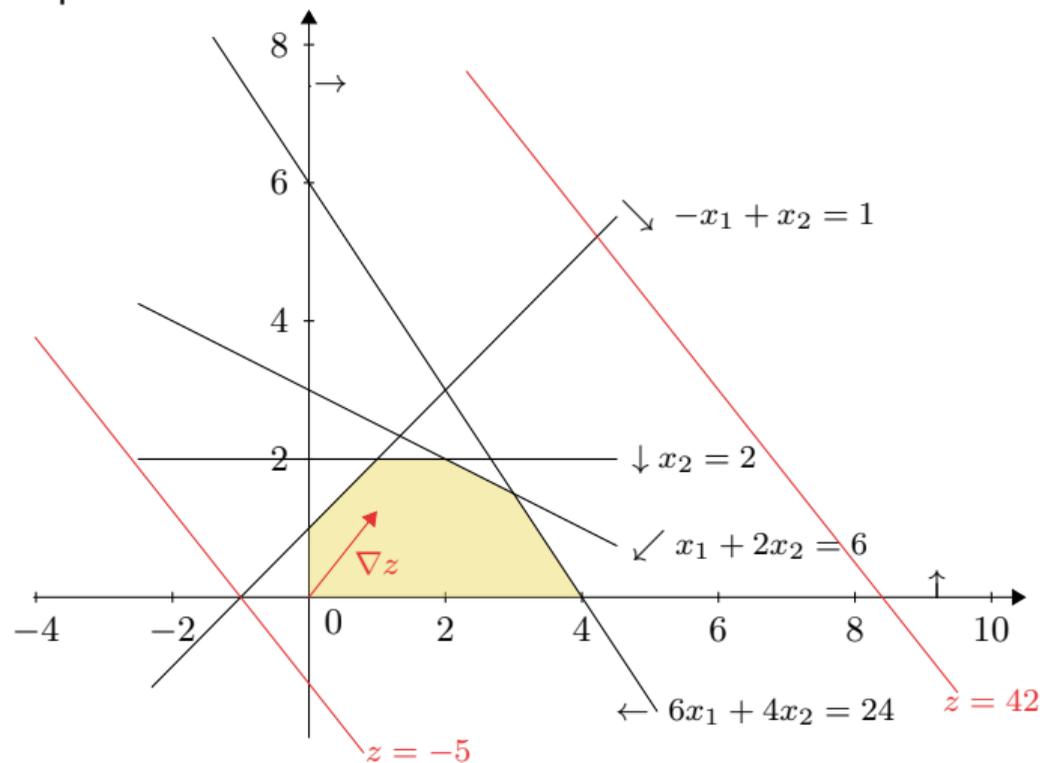& x_1, x_2 \geq 0
\end{aligned}
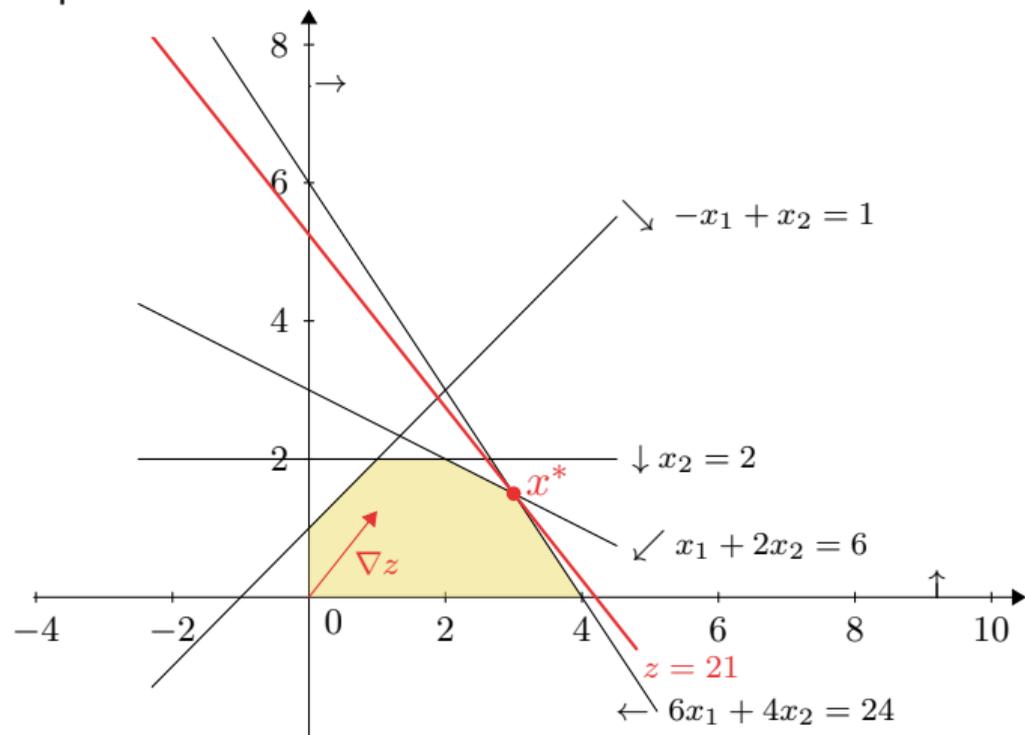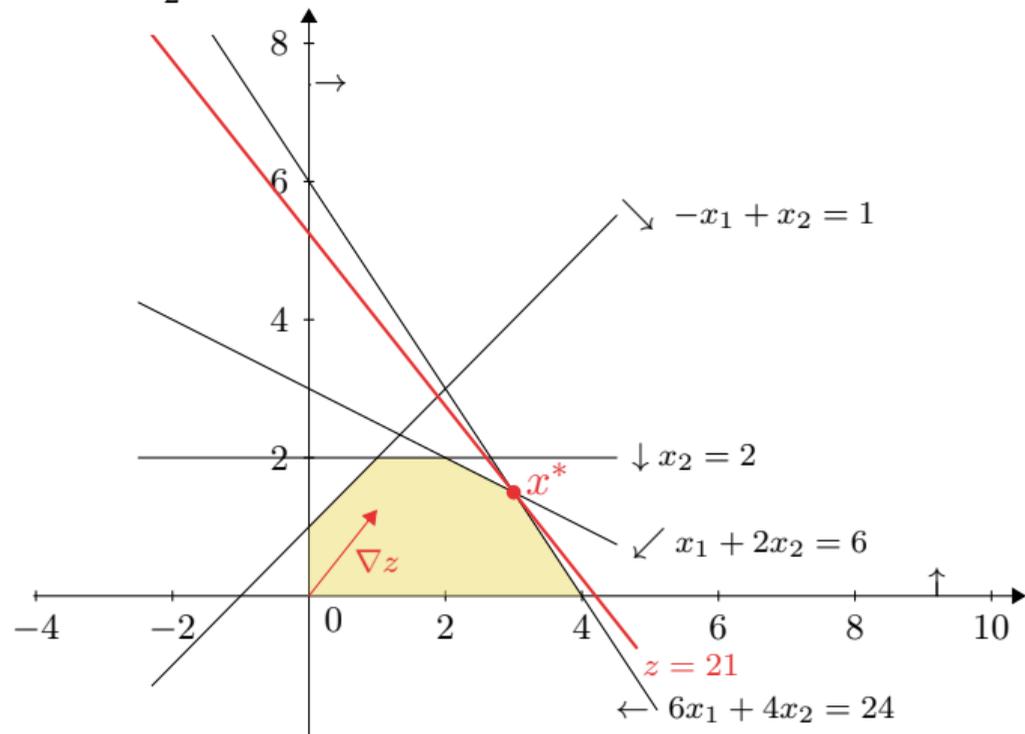$$

Example 1

Step 1
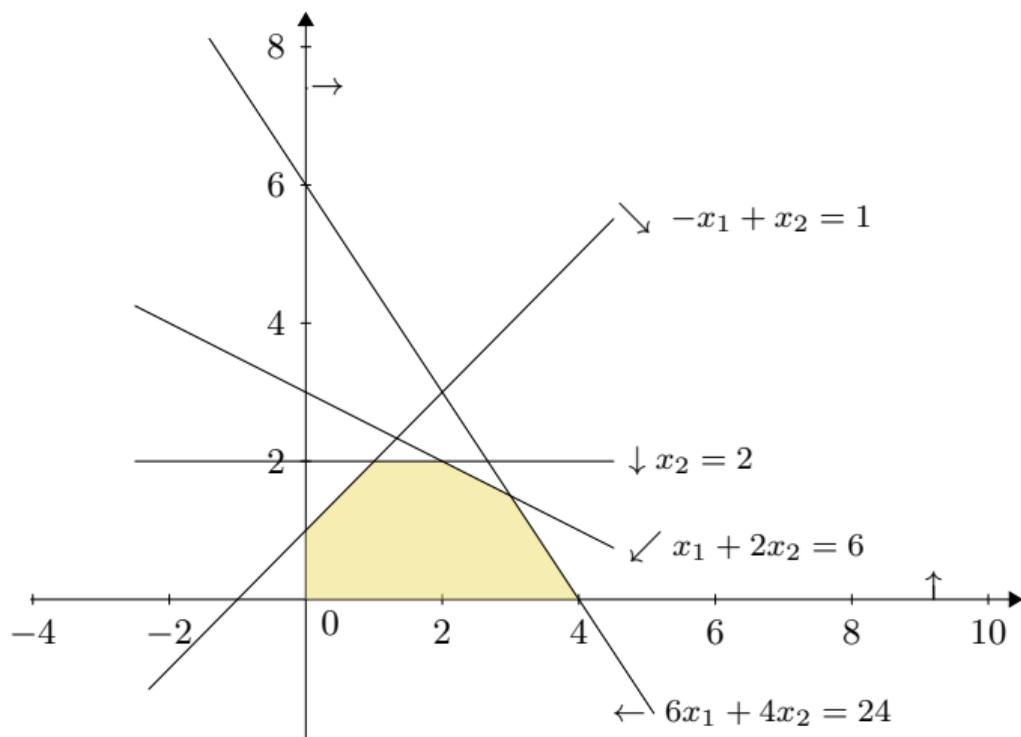
Example 1

Step 2

Example 1

Step 3

Example 1

unique optimal solution, the optimal solution is
$x^* = (3, \frac{3}{2}) = (3; 1.5)$ with value $z^* = 21$

Example 2

Use the graphical method to find the optimal solutions of the following LP.

$$\max \quad z = 5x_1 + 4x_2,$$

$$
\begin{aligned}
\text{s. to:} \quad & 6x_1 + 4x_2 \le 24, \\
& x_1 + 2x_2 \le 6, \\
& -x_1 + x_2 \le 1, \\
& x_2 \le 2, \\
& 2x_1 + x_2 \ge 9, \\
& x_1, x_2 \ge 0
\end{aligned}
$$

$-x_1 + x_2 = 1$

$\downarrow x_2 = 2$

$\swarrow x_1 + 2x_2 = 6$

$\leftarrow 6x_1 + 4x_2 = 24$

$\rightarrow 2x_1 + x_2 = 9$

Example 2

we obtain an empty feasible set, thus the problem is impossible

Example 3

Use the graphical method to identify the optimal solutions of the following LP.

$$\max \quad z = 5x_1 + 4x_2,$$

$$\text{s. to:} \quad -x_1 + x_2 \leq 1,$$
$$x_2 \leq 2,$$
$$x_1, x_2 \geq 0$$

Example 3

the feasible region is unbounded and the problem is unbounded

Example 4

Use the graphical method to identify the optimal solutions of the following LP.

$$\max \quad z = -x_1 + 2x_2,$$
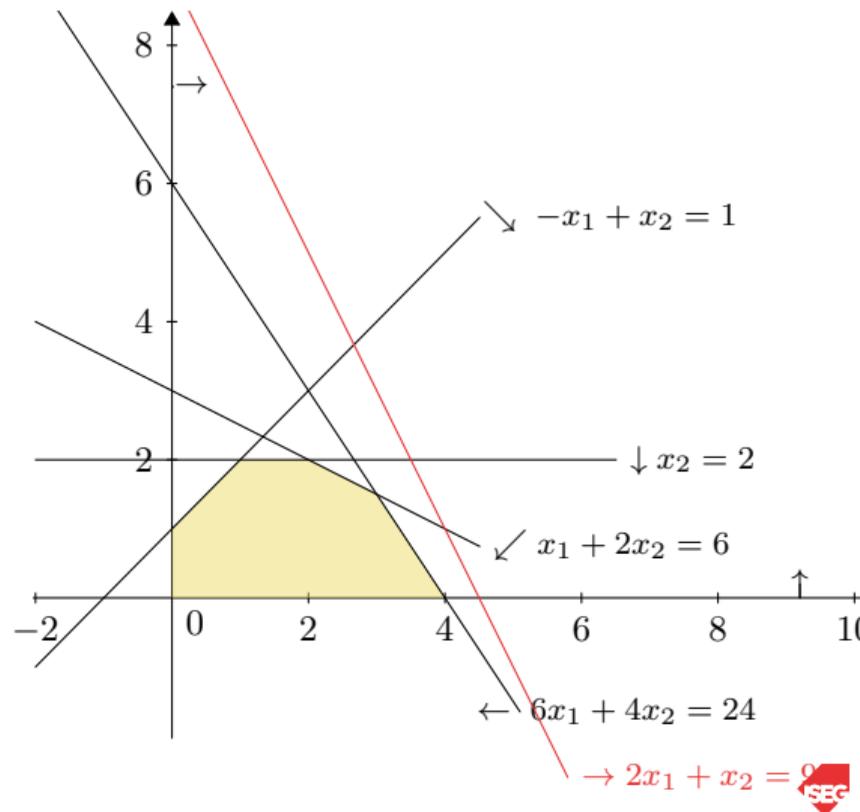
$$\text{s. to:} \quad -x_1 + x_2 \leq 1,$$
$$x_2 \leq 2,$$
$$x_1, x_2 \geq 0$$

Example 4

feasible region is unbounded and the unique optimal solution is $(1, 2)$

Example 5

Use the graphical method to identify the optimal solutions of the following LP.

$$\max \quad z = 6x_1 + 4x_2,$$

$$\text{s. to:} \quad 6x_1 + 4x_2 \leq 24,$$
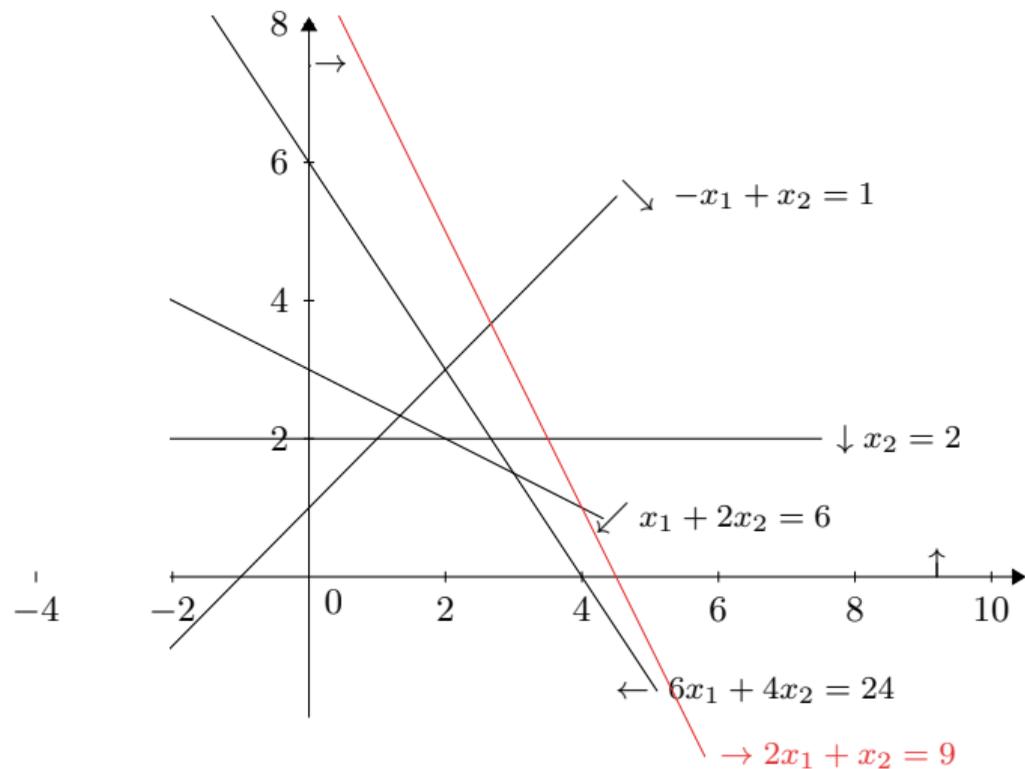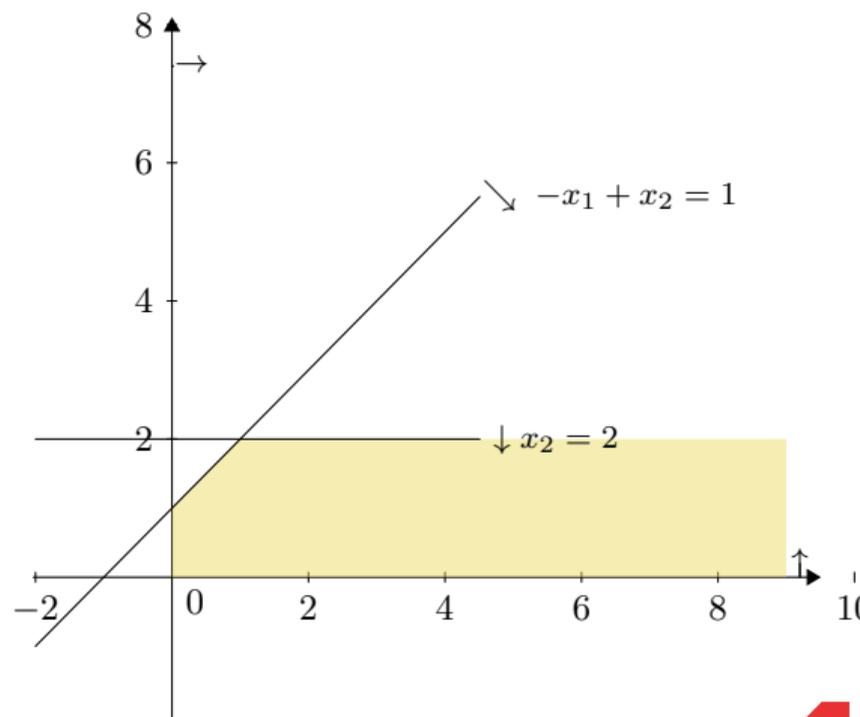$$x_1 + 2x_2 \leq 6,$$
$$-x_1 + x_2 \leq 1,$$
$$x_2 \leq 2,$$
$$x_1, x_2 \geq 0$$

$-x_1 + x_2 = 1$

$\downarrow x_2 = 2$

$x_1 + 2x_2 = 6$

∇z

$z = 24$

$\leftarrow 6x_1 + 4x_2 = 24$

Example 5

alternative optimal solutions $x^* = \alpha(3, \frac{3}{2}) + (1-\alpha)(4, 0),\ \alpha \in [0, 1]$, the line segment between

Example 6

Use the graphical method to identify the optimal solutions of the following LP.

$$\max \quad z = 2x_2,$$
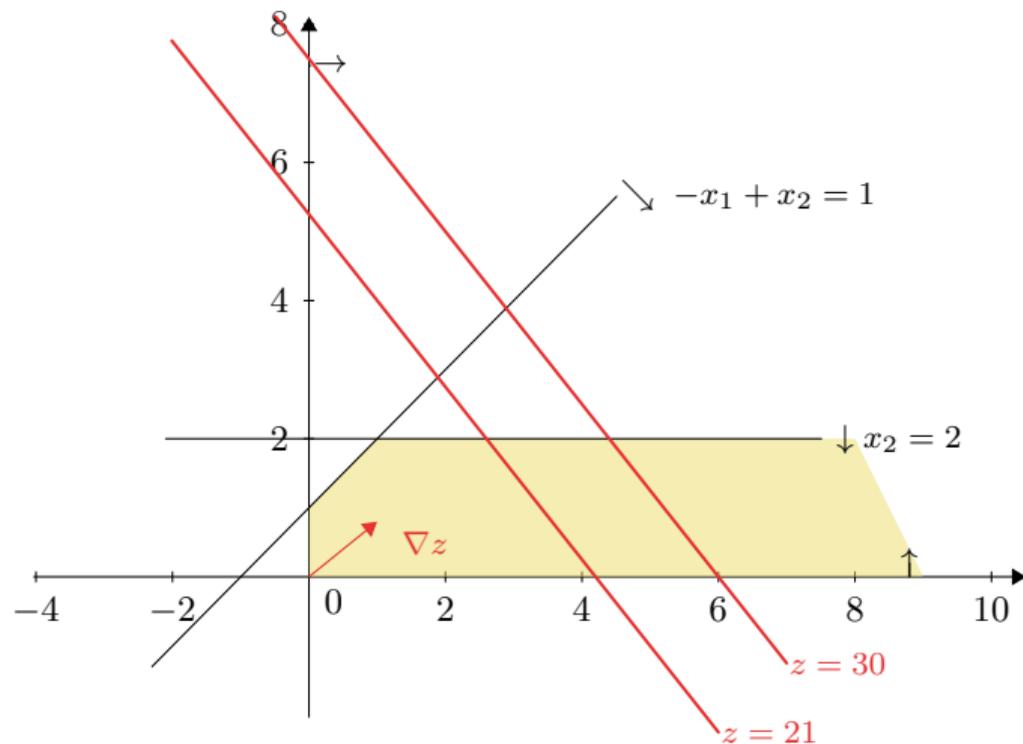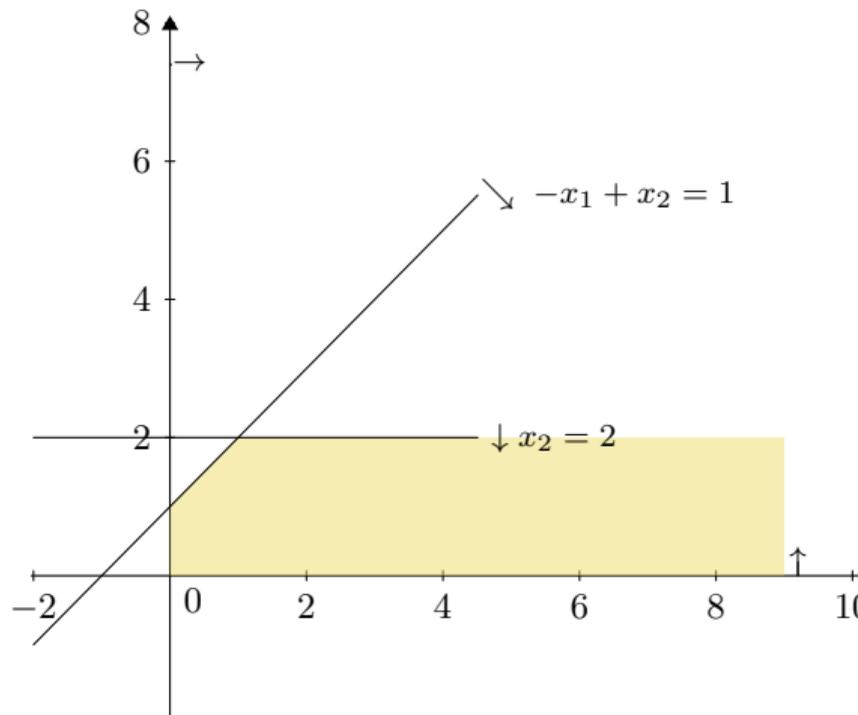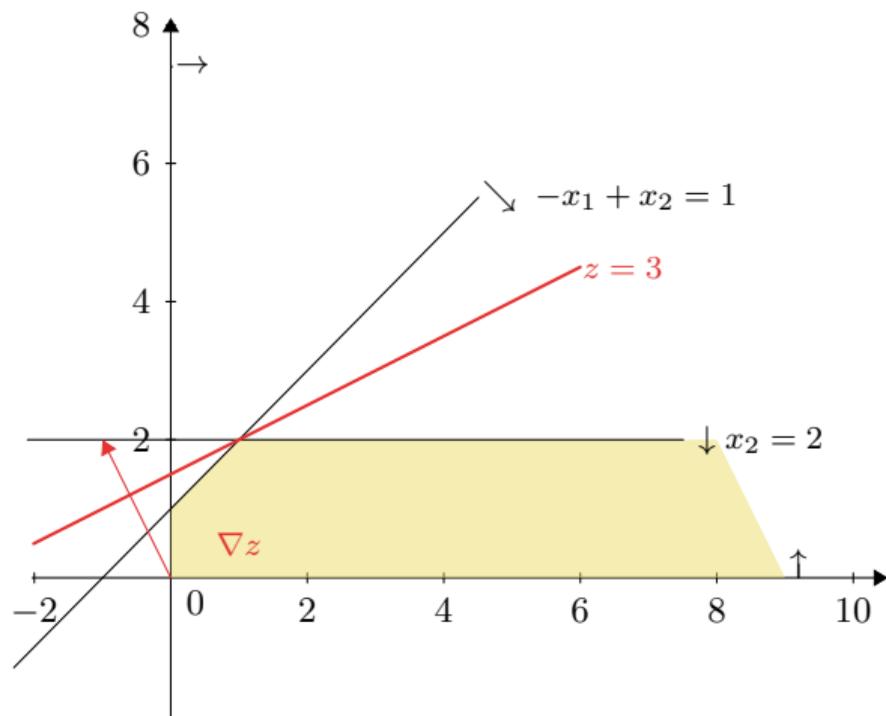
$$\text{s. to:} \quad -x_1 + x_2 \leq 1,$$
$$x_2 \leq 2,$$
$$x_1, x_2 \geq 0$$
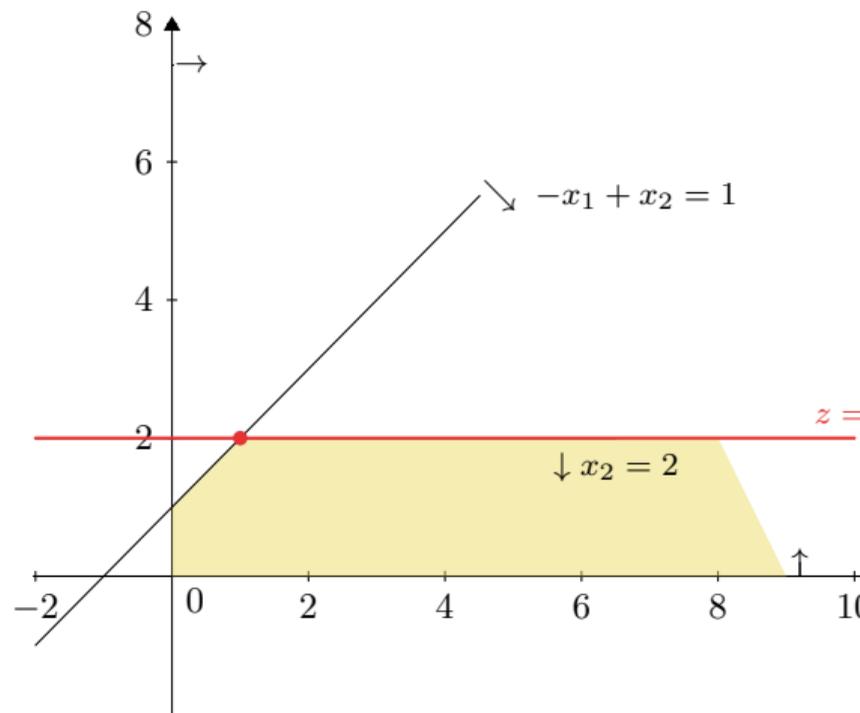
Example 6
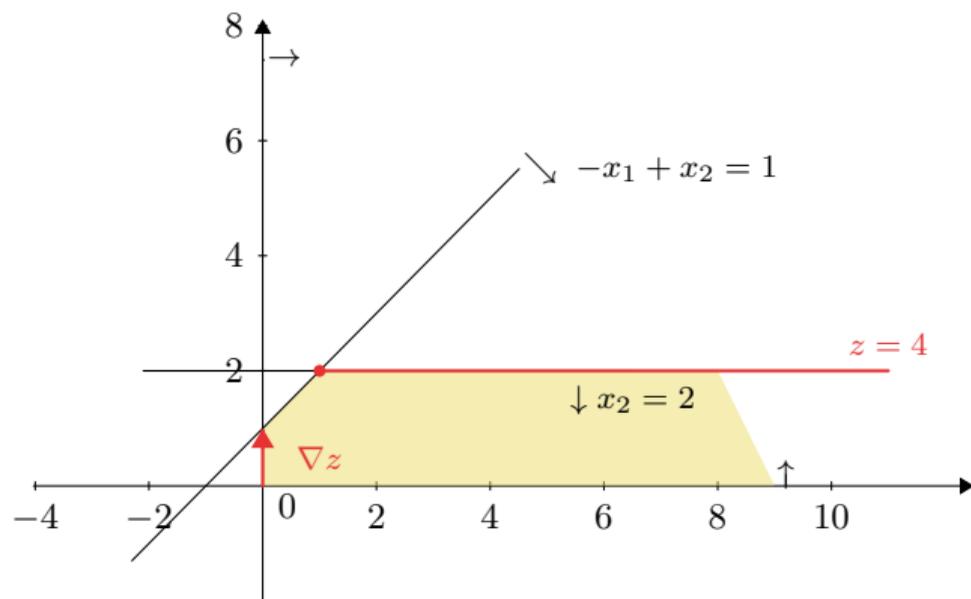
feasible region is unbounded and the alternative optimal solutions are
$x^* = (1,2) + \beta(1,0), \ \beta \geq 0$, the semi-line starting at point $(1,2)$ and direction of vector $(1,0)$

Example 7

Use the graphical method to identify the optimal solutions of the following LP.

$\max Z = x_1 + 3x_2$
s. to $\quad x_1 - 3x_2 \leq 3$
$\qquad -2x_1 + x_2 \leq 2$
$\qquad -3x_1 + 4x_2 \leq 12$
$\qquad 3x_1 + x_2 \leq 9$
$\qquad x_1, \ x_2 \geq 0$

Sketch the feasible region in the space of variables $\{x_1, x_2\}$ and identify the optimal solution.

Example 7
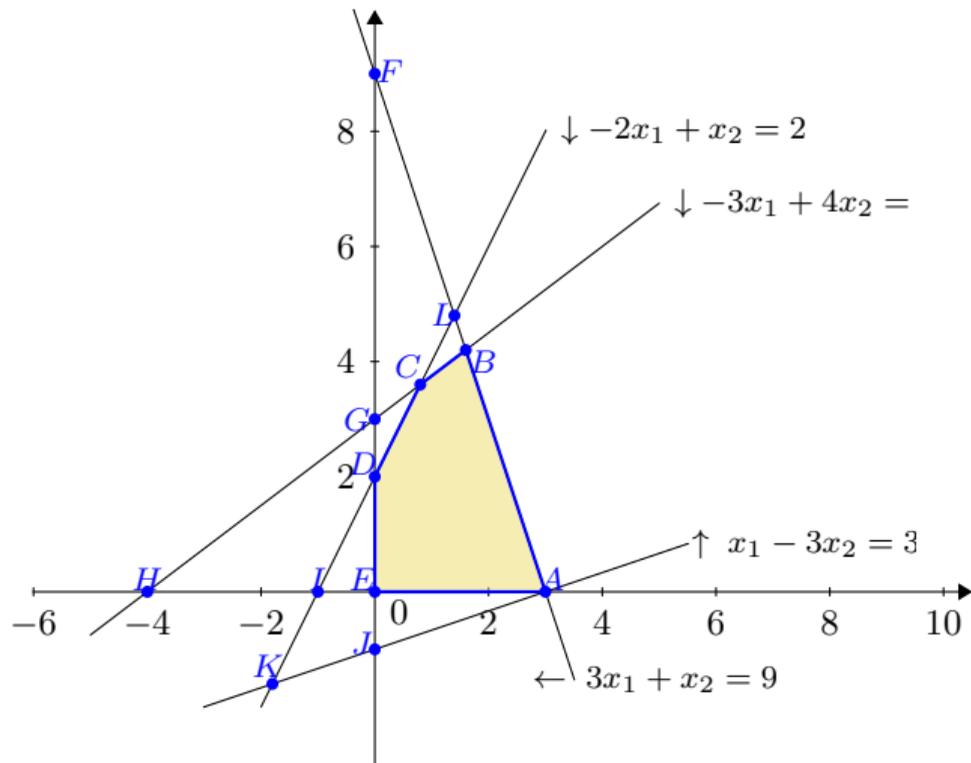
Example 7

the gradient of the objective function is the vector $\nabla Z = (1, 3)$
the unique optimal solution is B
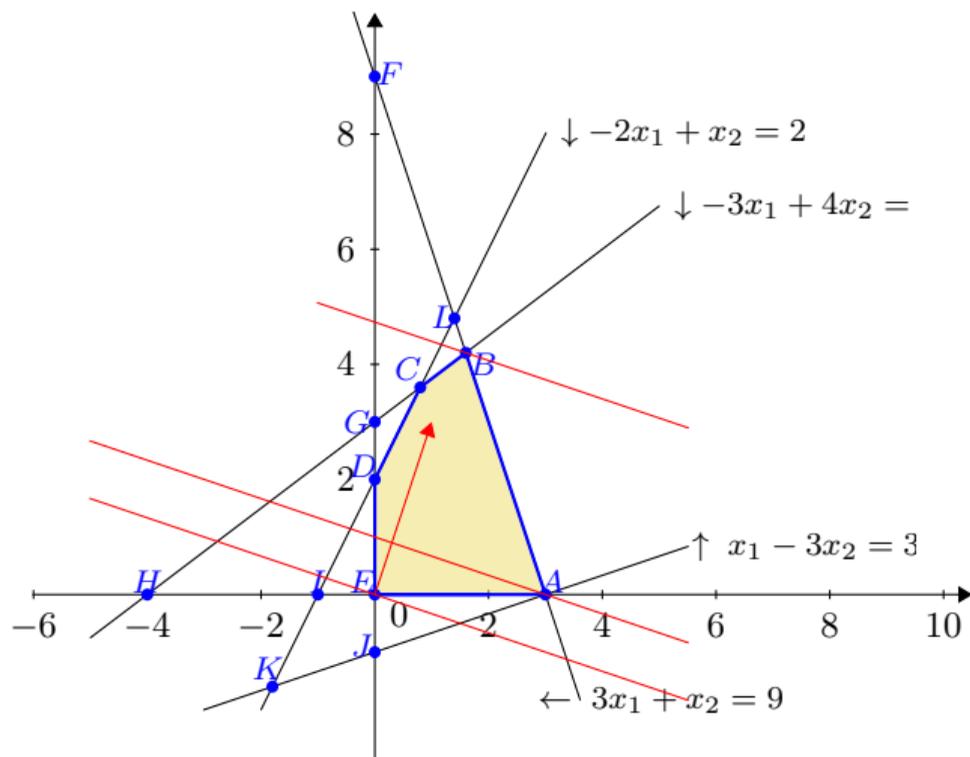
Solving LP models

Solve an LP: find optimal solutions, when exist

To solve an LP is to determine the optimal solution (or solutions) and the optimal value or to conclude that an optimal solution does not exist and why.

- If the feasible region is an empty set, then the problem is impossible (see Example 2).
- If the feasible region is a non-empty set, then the problem is possible,
  - if there is an optimal solution, find its value and all its (alternative) optimal solutions,
    - the problem can have a unique optimal solution (see Examples 1, 4, 7),
    - the problem can have alternative optimal solutions (in this case it has an infinite number of optimal solutions) (see Examples 5, 6),
  - otherwise, conclude that the problem has an objective function with unlimited value, the problem is unbounded (see Example 3).

Solve an LP: optimal solutions

a non-empty feasible region of an LP is always a convex set, it can be bounded or unbounded

types of optimal solutions of an LP problem

- ▶ unique optimal solution
  - ▶ is an extreme point of the feasible set
    (see examples 1, 4, 7)
- ▶ alternative optimal solutions
  - ▶ linear convex combination of extreme points of the feasible set
    (see example 5)
  - ▶ if the feasible set is unbounded, it can also be a linear convex combination of extreme points of the feasible set plus a linear positive combination of extreme directions of the feasible set
    (see example 6)

Solving with the Excel Solver

Excel Solver

The software Excel, with the Add-in Solver, can easily be used to solve small LP problems.

The first time, make sure you have the Solver add-in installed in the Excel software.

Go to: File (at the top left) $\rightarrow$ Options (at the bottom) $\rightarrow$ Add-ins (on the left panel) $\rightarrow$ Manage: Excel Add-ins (at the bottom) $\rightarrow$ Go $\rightarrow$ Select the Solver Add-in $\rightarrow$ OK.

Excel Solver

1. Identify a column for each decision variable.
2. Identify a row for the objective function and for each constraint.
3. Insert all the data:
   3.1 coefficients in the objective function, in the corresponding row and column associated to each variable;
   3.2 technological coefficients in each corresponding constraint row and column associated to each variable;
   3.3 for each constraint, also insert the corresponding RHS in a separate column.
4. Select the cells where you want the Excel Solver to display the optimal values of the decision variables.

Excel Solver

5. Select a cell where you want the Excel Solver to display the value of the optimal solution. Write in the cell, using the SUMPRODUCT function of the Excel, the objective function, using the cells of the coefficients and the cells selected to display the optimal values of the decision variables.

6. For each constraint, write in a separate cell, using the SUMPRODUCT function of the Excel, the constraint function, using the cells of the coefficients and the cells selected to display the optimal values of the decision variables.

Excel Solver

7. Go to the Data panel at the top, select the solver (that you have installed) from the available operations.

  7.1 Make sure the cell you selected to display the value of the optimal solution, corresponds to the one in the box Set Objective.

  7.2 Select the Max or Min criteria for your problem.

  7.3 Make sure the cells you selected to display the optimal value of the decision variables appear in the By Changing Variable Cells box.

  7.4 Use the Add botton to add all the constraints by selecting: the cell where the constraint function is written, the cell where the rhs is located, and select the sign of the inequality.

  7.5 Select a solving method: Simplex LP

  7.6 Press the Solve botton.

  7.7 Select the Reports and press OK.

Example of the Excel Solver

Use the Excel Solver to find the optimal solutions of the following LP.

$$\max \quad z = 4x_1 + 5x_2,$$

$$
\begin{aligned}
\text{s. to:} \quad & 4x_1 + 6x_2 \leq 24, \\
& 2x_1 + x_2 \leq 6, \\
& x_1 - x_2 \leq 1, \\
& x_1 \qquad \leq 2, \\
& x_1, x_2 \geq 0
\end{aligned}
$$

Example of the Excel Solver

Steps 1, 2, and 3

	A	B	C	D	E	F
1		x1	x2			
2	value					
3						
4	of	4	5			
5						
6	c1	4	6			24
7	c2	2	1			6
8	c3	1	-1			1
9	c4	1	0			2

Example of the Excel Solver

Steps 4, and 5

	A	B	C	D	E	F
1		x1	x2			
2	value					
3						
4	of	4	5	=SUMPRODUCT(B4:C4;B$2:C$2)		
5						
6	c1	4	6			24
7	c2	2	1			6
8	c3	1	-1			1
9	c4	1	0			2

SUM   =SUMPRODUCT(B4:C4;B$2:C$2)

Example of the Excel Solver

Step 6, copy the formula for all the constraints

Example of the Excel Solver

Step 7 - 1,2,3

Step 7 - 4

Example of the Excel Solver

Step 7 - 1,2,3,4

Example of the Excel Solver

Step 7 - 5, 6

Select a Solving Method: Simplex LP [Options]

Solving Method

Select the GRG Nonlinear engine for Solver Problems that are smooth nonlinear. Select the LP Simplex engine for linear Solver Problems, and select the Evolutionary engine for Solver problems that are non-smooth.

[Help] [Solve] [Close]

Example of the Excel Solver

Step 7 - after 6

Solver Results

Solver found a solution. All Constraints and optimality conditions are satisfied.

Reports

- ⦿ Keep Solver Solution
- ○ Restore Original Values

Answer
Sensitivity
Limits

☐ Return to Solver Parameters Dialog   ☐ Outline Reports

[OK]   [Cancel]   [Save Scenario...]

Solver found a solution. All Constraints and optimality conditions are satisfied.

When the GRG engine is used, Solver has found at least a local optimal solution. When Simplex LP is used, this means Solver has found a global optimal solution.

Example of the Excel Solver

Step 7 - 7

Solver Results   ×

Solver found a solution. All Constraints and
optimality conditions are satisfied.

- ● Keep Solver Solution

- ○ Restore Original Values

Reports

| Answer |
| Sensitivity |
| Limits |

☐ Return to Solver Parameters Dialog   ☐ Outline Reports

OK   Cancel   Save Scenario...

Reports

Creates the type of report that you specify, and places each report on a separate sheet in the workbook

Example of the Excel Solver

The results obtained

	A	B	C	D	E	F
1		x1	x2			
2	value	1,5	3			
3						
4	of	4	5	21		
5						
6	c1	4	6	24		24
7	c2	2	1	6		6
8	c3	1	-1	-1,5		1
9	c4	1	0	1,5		2
10						

Example of the Excel Solver

The Answer Report

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$D$4	of	0	21

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$2	value x1	0	1,5	Contin
$C$2	value x2	0	3	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$D$6	c1	24	$D$6<=$F$6	Binding	0
$D$7	c2	6	$D$7<=$F$7	Binding	0
$D$8	c3	-1,5	$D$8<=$F$8	Not Binding	2,5
$D$9	c4	1,5	$D$9<=$F$9	Not Binding	0,5

Example of the Excel Solver

The Sensitivity Report

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$2	value x1	1,5	0	4	6	0,666666667
$C$2	value x2	3	0	5	1	3

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$D$6	c1	24	0,75	24	12	4
$D$7	c2	6	0,5	6	0,666666667	2
$D$8	c3	-1,5	0	1	1E+30	2,5
$D$9	c4	1,5	0	2	1E+30	0,5

Solving an LP: explore several solvers

There are several methods to solve an LP

- ▶ Graphical method (only for models with two variables)
- ▶ Simplex method
- ▶ Interior points methods
- ▶ . . .

There are several software programs that use these methods to solve an LP:

- ▶ EXCEL with the Add-in Solver (only for small models)
- ▶ solvers such as XPRESS, CPLEX, Gurobi, COIN-OR CLP/CBC, GLPK, MOSEK, CHOCO, MIPCL, HiGHS, SCIP/FSCIP
- ▶ Languages that help the interaction with solvers Julia, Pyomo
- ▶ modules that help the interaction with solvers such as PuLP a modeler written in Python
- ▶ . . .

Definitions of LP models

Definitions

Solution of an LP: a vector of $\mathbb{R}^n$ which components are the values of the decision variables

Feasible Solution (FS): a solution that satisfies all the constraints (functional and sign)

Non Feasible Solution (NFS): a solution that does not satisfy at least one of the constraints

Feasible Region (FR): the set of all feasible solutions

Binding constraint in a solution: a constraint that holds with equality at that solution

Optimal Solution (OS): a feasible solution that gives the best value to the objective function (OF) (the best value=maximum or minimum)

Optimal value: the value of the objective function at an optimal solution

Exercises

1. Model and solve

Dorian Auto produces luxury cars and caravans, targeting high-income men and women. It was decided to launch a TV advertising campaign during comedy shows and football matches.

Each comedy show is watched by 7 million High-Income Women (HIW) and 2 million High-Income Men (HIM). Each football match is watched by 2 million HIW and 12 million HIM.

Each minute of advertising during comedy shows costs $50,000, and during football matches costs $100,000.

Dorian Auto wants the ads to be seen by at least 28 million HIW and 24 million HIM. Use Linear Programming to determine how Dorian Auto should run the campaign while meeting these requirements at the lowest possible cost.

2. Model and solve

Ozark Farms uses at least 800 Kg of special feed daily. The special feed is a mixture of corn and soybean meal with the following compositions:

feedstuff	protein (Kg)	fiber (Kg)	Cost ($/Kg)
Corn	0.09	0.02	0.30
Soybean meal	0.60	0.06	0.90

Tabela: Components (Kg) and Cost ($) per Kg of feedstuff

The dietary requirements of special feed are at least 30% protein and at most 5% fiber. The goal is to determine the daily minimum-cost feed mix.

Model

x_1 Kg of corn in the daily mix

x_2 Kg of soybean meal in the daily mix

$$\min \quad z = 0.3x_1 + 0.9x_2, \quad \leftarrow \text{total daily cost (\$)}$$

$$
\begin{aligned}
\text{s. to:} \quad & x_1 + x_2 \geq 800, && \leftarrow \text{Kg of feed required daily} \\
& 0.21x_1 - 0.3x_2 \leq 0, && \leftarrow \text{protein requirement, min 30\%} \\
& 0.03x_1 - 0.01x_2 \geq 0, && \leftarrow \text{fibber requirement, max 5\%} \\
& x_1, x_2 \geq 0, && \leftarrow \text{no negative amounts}
\end{aligned}
$$
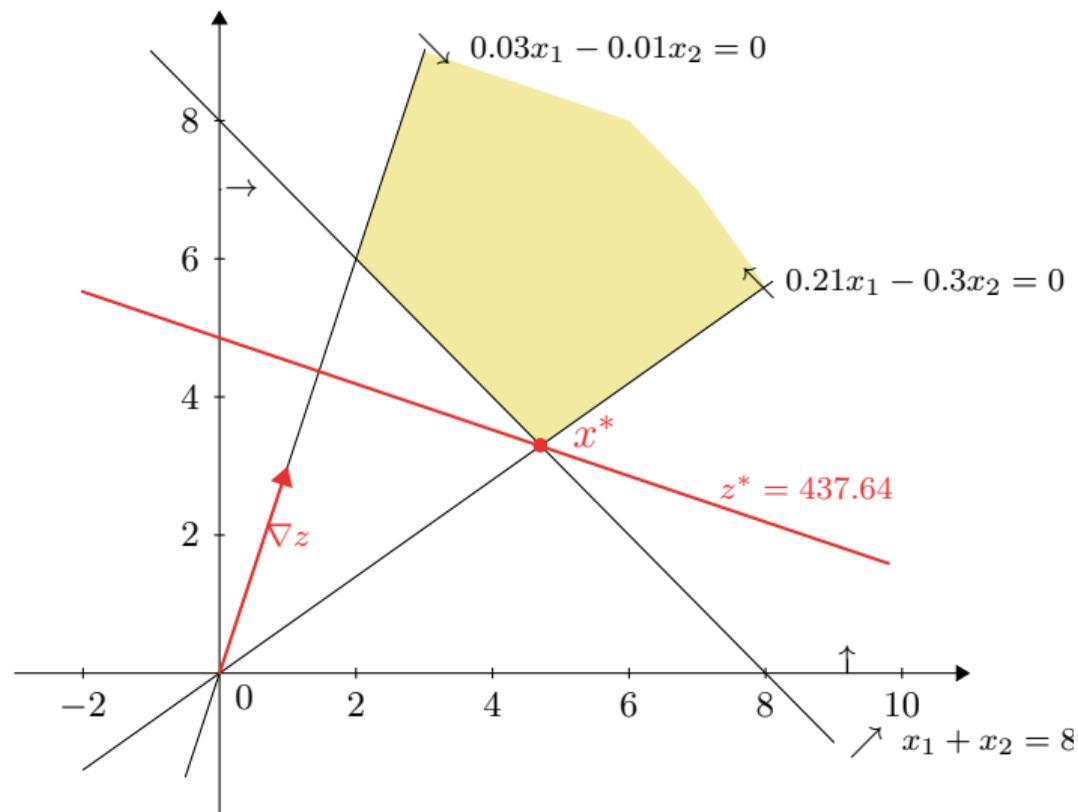
$x^* = (470.6; 329.4)$ and $z^* = 437.64$

Answer Report from Excel Solver

Objective Cell (Min)

Cell	Name	Original Value	Final Value
$E$4	z	0	437,647

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$C$10	x1	0	470,588	Contin
$D$10	x2	0	329,412	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$E$6	C1	800	$E$6>=$G$6	Binding	0
$E$7	C2	-1,429E-14	$E$7>=$G$7	Binding	0
$E$8	C3	10,824	$E$8>=$G$8	Not Binding	10,824

Assumptions of LP models

Assumptions of LP models

Proportionality: The contribution of each activity j to the value of the objective function and to the left-hand-side of the constraints is proportional to the level x_j of the activity.

Additivity: The value of the objective function and the value of the left-hand-side of the constraints are the sum of the individual contributions of the various activities.

Divisibility: The decision variables $x_j \in \mathbb{R}$ assume real values.

Certainty: Every coefficient (also called parameter) is assumed to be a known constant.

Representing LP models

General Form of a Linear Programming Problem

Let $x_j \in \mathbb{R}$, $j \in J = J_1 \cup J_2 \cup J_3$, denote the decision variables

$$\text{minimize or maximize} \quad z = \sum_{j \in J} c_j x_j$$

$$\text{subject to} \quad \sum_{j \in J} a_{ij} x_j \leq b_i, \quad i \in I_1, \quad (1)$$

$$\sum_{j \in J} a_{ij} x_j \geq b_i, \quad i \in I_2, \quad (2)$$

$$\sum_{j \in J} a_{ij} x_j = b_i, \quad i \in I_3, \quad (3)$$

$$x_j \geq 0, \qquad j \in J_1, \quad (4)$$
$$x_j \leq 0, \qquad j \in J_2, \quad (5)$$
$$x_j \in \mathbb{R}, \qquad j \in J_3, \quad (6)$$

Parameters

▶ c_j: cost coefficient of decision variable x_j in the objective function, $c_j \in \mathbb{R}$, $j \in J$,

▶ a_{ij}: technological coefficient of decision variable x_j in constraint i, $i \in I$, $j \in J$,

▶ b_i: right-hand side (RHS) of functional constraint i, $b_i \in \mathbb{R}$, $i \in I = I_1 \cup I_2 \cup I_3$.

General Form of a Linear Programming Problem

Let $x \in \mathbb{R}^{|J|}$, $J = J_1 \cup J_2 \cup J_3$, denote the vector of decision variables

$$\max / \min \quad z = \sum_{j \in J} c_j x_j$$

$$\text{s. t.:} \quad \sum_{j \in J} a_{ij} x_j \begin{cases} \leq \\ \geq \\ = \end{cases} b_i, \ i \in I$$

$$x_j \geq 0, j \in J_1, x_j \leq 0, j \in J_2, \ x_j \in \mathbb{R}, j \in J_3$$

with

- $x = (x_1, x_2, \ldots, x_{\bar{n}}) \in \mathbb{R}^{\bar{n}}$, $|J| = \bar{n}$ decision variables,
- $c = (c_1, c_2, \ldots, c_{\bar{n}}) \in \mathbb{R}^{\bar{n}}$ cost coefficients,
- m (technological) constraints, $|J| = m$,
- $\bar{A}_{m \times \bar{n}}$ matrix $\bar{A} = [a_{ij}]$ technological coefficients,
- $b = (b_1, b_2, \ldots, b_m) \in \mathbb{R}^m$ RHS vector

Canonical Maximization LP Form

A LP with $\bar{n}$ variables and m constraints is in the max canonical form when

$$\max \qquad z = \sum_{j=1}^{\bar{n}} c_j x_j$$

s. to:
$$\sum_{j=1}^{\bar{n}} a_{ij} x_j \leq b_i, \quad i = 1, \ldots, m$$
$$x_j \geq 0, \qquad \qquad j = 1, \ldots, \bar{n}$$

Canonical Minimization LP Form

A LP with $\bar{n}$ variables and m constraints is in the min canonical form when

$$\min \quad z = \sum_{j=1}^{\bar{n}} c_j x_j$$

s. to:

$$\sum_{j=1}^{\bar{n}} a_{ij} x_j \geq b_i, \quad i = 1, \ldots, m$$

$$x_j \geq 0, \qquad j = 1, \ldots, \bar{n}$$

A LP with $\bar{n}$ variables and m constraints is in the standard form when

$$\min/\max \quad z = \sum_{j=1}^{\bar{n}} c_j x_j$$

s. t.:
$$\sum_{j=1}^{\bar{n}} a_{ij} x_j = b_i, \quad i = 1, \ldots, m$$
$$x_j \geq 0, \qquad \quad j = 1, \ldots, \bar{n}$$

Property

Theorem

Any LP in the general form can be transformed into an equivalent problem in the canonical form and also in the standard form.

Reformulation operations on constraints

an inequality can be transformed into an opposite, in sign, inequality

$$\sum_{j=1}^{n} a_{ij}x_j \geq b_i \qquad \Leftrightarrow \qquad \sum_{j=1}^{n} -a_{ij}x_j \leq -b_i$$

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i \qquad \Leftrightarrow \qquad \sum_{j=1}^{n} -a_{ij}x_j \geq -b_i$$

an equality is always equivalent to two inequalities

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \qquad \Leftrightarrow \qquad \begin{cases} \sum_{j=1}^{n} a_{ij}x_j \leq b_i \\ \sum_{j=1}^{n} a_{ij}x_j \geq b_i \end{cases}$$

Reformulation operations on constraints

an inequality can be transformed into an equality by including a non-negative slack variable, $x_i^s \geq 0$,

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i \qquad \Leftrightarrow \qquad \begin{cases} \sum_{j=1}^{n} a_{ij} x_j - x_i^s = b_i \\ x_i^s \geq 0 \end{cases}$$

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad \Leftrightarrow \qquad \begin{cases} \sum_{j=1}^{n} a_{ij} x_j + x_i^s = b_i \\ x_i^s \geq 0 \end{cases}$$

Reformulation operations on the domain constraints

usually the domain constraints of the variables are non-negativity constraints, replace variable x_j by x_j' (and x_j'')

$$x_j \leq 0 \quad \rightarrow \quad \begin{cases} x_j = -x_j' \\ x_j' \geq 0 \end{cases}$$

$$x_j \in \mathbb{R} \quad \rightarrow \quad \begin{cases} x_j = x_j' - x_j'' \\ x_j', x_j'' \geq 0 \end{cases}$$

$$x_j \geq l_j \quad \rightarrow \quad \begin{cases} x_j' = x_j - l_j \\ x_j' \geq 0 \end{cases}$$

$$x_j \leq u_j \quad \rightarrow \quad \begin{cases} x_j' = u_j - x_j \\ x_j' \geq 0 \end{cases}$$

$$max \; z = -min - z$$

$$min \; z = -max - z$$

Example of a standard form: the RM model

Consider the non-negative slack variables $x_3, x_4, x_5, x_6 \geq 0$, the standard form of the RM model is

$$\max \quad z = 4x_1 + 5x_2,$$

$$
\begin{aligned}
\text{s. t.:} \quad 4x_1 + 6x_2 + x_3 \phantom{{}+ x_4 + x_5 + x_6} &= 24, \\
2x_1 + x_2 \phantom{{}+ 6x_2} + x_4 \phantom{{}+ x_5 + x_6} &= 6, \\
x_1 - x_2 \phantom{{}+ x_4} + x_5 \phantom{{}+ x_6} &= 1, \\
x_1 \phantom{{}- x_2 + x_4 + x_5} + x_6 &= 2, \\
x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0
\end{aligned}
$$

Some convexity results

Convex linear combination

Definition

A vector $x \in \mathbb{R}^n$ is a **linear combination** of the vectors $x_1, \ldots, x_k \in \mathbb{R}^n$ if, for some $\lambda \in \mathbb{R}^k$, we have

$$x = \sum_{i=1}^{k} \lambda_i x_i$$

If, in addition, $\lambda_i \geq 0$, $i = 1, \ldots, k$, and $\sum_{i=1}^{k} \lambda_i = 1$, then x is a **convex linear combination** of the vectors $x_1, \ldots, x_k$.

Notation

For a nonempty subset $S \subseteq \mathbb{R}^n$, $conv(S)$ denotes the **convex hull** of the elements of S, that is, the set of all vectors that are finite convex combinations of vectors in S.

Convex set

Definition

A subset $S \subseteq \mathbb{R}^n$ is a **convex set** if $S = conv(S)$.

Alternative definition

A subset $S \subseteq \mathbb{R}^n$ is convex if
$\forall x_1, x_2 \in S$ and $\lambda \in [0,1]$,

$$\lambda x_1 + (1 - \lambda)x_2 \in S.$$

Properties

- $conv(S)$ is the smallest convex set that contains S.
- $conv(S)$ is the intersection of all convex sets that contain S.
- If $S_1, S_2 \subseteq \mathbb{R}^n$ are convex, then $S_1 \cap S_2$ is convex.

Simplex

A set of points $x_1, \ldots, x_k \in \mathbb{R}^n$ is said to be **affinely independent** if the unique solution of

$$\sum_{i=1}^{k} \lambda_i x_i = 0, \qquad \sum_{i=1}^{k} \lambda_i = 0$$

is $\lambda_i = 0$, $i = 1, \ldots, k$.

Proposition

The points $x_1, \ldots, x_k \in \mathbb{R}^n$ are affinely independent if and only if

$$x_2 - x_1, \ldots, x_k - x_1 \in \mathbb{R}^n$$

are linearly independent.

Definition

The convex hull of a finite number of affinely independent points $x_1, x_2, \ldots, x_k \in \mathbb{R}^n$ is called a simplex with vertices $x_1, x_2, \ldots, x_k$.

Polytope and Polyhedron

Definition

The convex hull of a finite number of points is called a **polytope**.

Definition

A polyhedron $X \subseteq \mathbb{R}^n$ is the set of points that satisfy a finite number of linear inequalities.

$$X = \{x \in \mathbb{R}^n : Ax \leq b\}$$

Properties

- A polyhedron is a convex set.
- A bounded polyhedron is a polytope.

Extreme points and extreme rays

Definitions

- $x \in X$ is an **extreme point** of X if there do not exist $x^1, x^2 \in X$, with $x^1 \neq x^2$, such that
$$x = \frac{1}{2}x^1 + \frac{1}{2}x^2.$$

- $r \in \mathbb{R}^n$ is a **ray** of X if and only if for any point $x \in X$, the set
$$\{y \in \mathbb{R}^n : y = x + \lambda r, \lambda \geq 0\} \subseteq X.$$

- A ray r of X is an **extreme ray** of X if $r = \frac{1}{2}r^1 + \frac{1}{2}r^2$, where r^1 and r^2 are rays of X, implies that $r^1 = \lambda_1 r$, $\lambda_1 \geq 0$, and $r^2 = \lambda_2 r$, $\lambda_2 \geq 0$.

Property

A polyhedron has a finite number of extreme points and extreme rays.

Minkowski's Theorem

Minkowski's Theorem

If $X = \{x \in \mathbb{R}^n : Ax \leq b\} \neq \emptyset$ and $rank(A) = n$, then

$$X = \{x \in \mathbb{R}^n : x = \sum_{k \in K} \lambda_k x^k + \sum_{j \in J} \mu_j r^j :$$
$$\sum_{k \in K} \lambda_k = 1, \lambda_k \geq 0, k \in K, \mu_j \geq 0, j \in J\}$$

where $E = \{x^k, k \in K\}$ is the set of extreme points and $R = \{r^j, j \in J\}$ is the set of extreme rays.

Feasible solution set

Property

The set $X = \{x \in \mathbb{R}^n : Ax \leq b, \ x \geq 0\}$ of feasible solutions of a linear programming problem is a polyhedron.

Therefore:

1. X is convex.
2. X has a finite number of extreme points and extreme rays.

Some convex sets in $\mathbb{R}^n$

Line

Given two distinct points $P, Q \in \mathbb{R}^n$, the **line** through P and Q is defined by

$$\ell(P, Q) = \{X \in \mathbb{R}^n : X = \lambda P + (1 - \lambda)Q, \ \lambda \in \mathbb{R}\}.$$

Line segment

Given two points $P, Q \in \mathbb{R}^n$, the **line segment** $\overline{PQ}$ is defined by

$$\overline{PQ} = \{X \in \mathbb{R}^n : X = \lambda P + (1 - \lambda)Q, \ \lambda \in [0, 1]\}.$$

Semi-line (Ray)

Given two distinct points $P, Q \in \mathbb{R}^n$, the **semi-line** (or **ray**) starting at Q and passing through P is defined by

$$\overrightarrow{QP} = \{X \in \mathbb{R}^n : X = \lambda P + (1 - \lambda)Q, \ \lambda \geq 0\}.$$

Some convex sets in $\mathbb{R}^n$

Hyperplane

Given a nonzero vector $a \in \mathbb{R}^n$ and a scalar $b \in \mathbb{R}$, a **hyperplane** is the set

$$H = \{x \in \mathbb{R}^n : a^\top x = b\}.$$

Semispace (Halfspace)

Given a nonzero vector $a \in \mathbb{R}^n$ and a scalar $b \in \mathbb{R}$, a **semispace** (or **halfspace**) is defined as

$$H^- = \{x \in \mathbb{R}^n : a^\top x \leq b\} \quad \text{or} \quad H^+ = \{x \in \mathbb{R}^n : a^\top x \geq b\}.$$