Mathematical Programming I

BSc in Applied Mathematics for Economics and Management (MAEG)

Lisbon School of Economics & Management
Universidade de Lisboa

2025-2026

Pos-optimization and Sensitivity analysis

Motivation

- After solving a linear program (LP), what happens if the data changes slightly?
- Can we reuse the solution of the original problem to solve the new one efficiently?

Post-optimality (or re-optimization) consists of:

- Using information from the original optimal solution
- To quickly solve a modified version of the problem

Topics to be studied:

- Discrete changes in:
  - objective function coefficients
  - right-hand side values of constraints
- Introduction of:
  - a new variable
  - a new constraint
- Sensitivity analysis with respect to:
  - objective function coefficients
  - right-hand side values

I – Discrete changes in the objective function coefficients

the cost of one or more variables is modified from c_k to $\bar{c}_k$

1. if x_k is a nonbasic variable

   the vector c_B of the costs of the basic variables does not change, therefore $z_j = c_B B^{-1} A_j$ remains unchanged for all variables; replace the reduced cost $z_k - c_k \leq 0$ by the reduced cost

$$z_k - \bar{c}_k = z_k - c_k + (c_k - \bar{c}_k),$$

   - if $z_k - \bar{c}_k \leq 0$ the solution remains optimal,
   - if $z_k - \bar{c}_k > 0$ then x_k must enter the basis, proceeding as in the usual simplex algorithm

I – Discrete changes in the objective function coefficients, c

2. if x_k is a basic variable

the vector c_B of the costs of the basic variables is modified,
therefore $z_j = c_B B^{-1} A_j$ changes for all nonbasic variables;
it is necessary to update the

- reduced costs $z_j - c_j$,
- value of the objective function $c_B B^{-1} b$;

after updating the last row of the simplex table (reduced costs and objective function value), if optimality is no longer satisfied, proceed with the simplex algorithm as usual

Example 1

Formulation and the corresponding optimal table

$$\begin{aligned} \max \quad & z = 2x_1 + x_2 - x_3 \\ \text{s. a:} \quad & \\ & x_1 + 2x_2 + x_3 \leq 8 \\ & -x_1 + x_2 - 2x_3 \leq 4 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

x_B	x_1	x_2	x_3	x_4	x_5	$\bar{b}$
x_1	1	2	1	1	0	8
x_5	0	3	-1	1	1	12
$z_j - c_j$	0	3	3	2	0	16

Without solving the problem from the beginning, obtain a new optimal solution when the initial data of the problem are changed.

- 1. (a) $\bar{c}_2 = 5$ (b) $\bar{c}_2 = 2$ (c) $\bar{c}_2 = -2$
- 2. (a) $\bar{c}_1 = 1$ (b) $\bar{c}_1 = -2$

Example 2 – ToyCo Example (Taha)

Problem

ToyCo assembles three types of toys: trains, trucks, and cars.

Each day, 430, 460, and 420 minutes are available for assembly operations M1, M2, and M3, respectively.

The revenue per unit is $3, $2, and $5 for trains, trucks, and cars, respectively.

Each train requires 1, 3, and 1 minutes for operations M1, M2, and M3.

For trucks and cars, the required times are 2, 0, 4 and 1, 2, 0, respectively (0 means that operation is not required).

Example 2 – ToyCo Example (Taha)

Problem description

- ToyCo assembles three types of toys:
  - Trains (x_1), Trucks (x_2), Cars (x_3)
- Daily available time (in minutes):
  - Operation M_1: 430
  - Operation M_2: 460
  - Operation M_3: 420
- Unit revenue:

$$3 \text{ (train)}, \quad 2 \text{ (truck)}, \quad 5 \text{ (car)}$$

- Processing times (minutes per unit):

	M_1	M_2	M_3
Train	1	3	1
Truck	2	0	4
Car	1	2	0

Goal:

- Determine how many units of each toy to produce in order to **maximize total revenue**

Example 2 – ToyCo Example (Taha)

Mathematical Formulation:

- x_1 number of Trains to assemble,
- x_2 number of Trucks to assemble,
- x_3 number of Cars to assemble,

$$
\begin{aligned}
\text{max} \quad & z = 3x_1 + 2x_2 + 5x_3 \\
\text{s.t.} \quad & x_1 + 2x_2 + x_3 \leq 430 \quad (M_1) \\
& 3x_1 + 0x_2 + 2x_3 \leq 460 \quad (M_2) \\
& x_1 + 4x_2 + 0x_3 \leq 420 \quad (M_3) \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

Indicate the optimal production plan.

Example 2 – ToyCo Example (Taha)

Final Simplex table – Optimal table

x_B	x_1	x_2	x_3	x_4	x_5	x_6	$\bar{b}$
x_2	-1/4	1	0	1/2	-1/4	0	100
x_3	3/2	0	1	0	1/2	0	230
x_6	2	0	0	-2	1	1	20
$z_j - c_j$	4	0	0	1	2	0	1350

Observations:

- All $\bar{b} \geq 0 \Rightarrow$ primal feasible
- All $z_j - c_j \geq 0$ (for max problem) $\Rightarrow$ optimal

Example 2 – ToyCo Example (Taha)

Optimal Solution – Optimal production plan:

$$x_1 = 0, \quad x_2 = 100, \quad x_3 = 230,$$

Maximum revenue:

$$z^* = 3(0) + 2(100) + 5(230) = 0 + 200 + 1150 = 1350$$

Resource usage:

- M_1: $0 + 2(100) + 230 = 430 \leq 430$ $\Rightarrow$ binding
- M_2: $3(0) + 2(230) = 460$ $\Rightarrow$ binding
- M_3: $0 + 4(100) = 400 \leq 420$ (slack 20)

Example 2 – ToyCo Example (Taha)

New situation – Change in Selling Prices

The unit selling prices change from $(3, 2, 5)$ to $(6, 8, 3)$ for:

- Trains (x_1)
- Trucks (x_2)
- Cars (x_3)

Question:

- What happens to the optimal production plan?
- What happens to the total revenue?

Methodological approach:

- This is a **change in the objective function coefficients**.
- Check the current basis using **reduced costs**.
- If optimality conditions are violated, apply the **primal simplex method**.

II – Discrete changes in the constraint matrix A

II – Discrete changes in the constraint matrix A

the column of a variable in the constraint matrix is modified from A_k to $\bar{A}_k$

1. if x_k is a nonbasic variable

   the new column $B^{-1}\bar{A}_k$ must be computed, as well as the new reduced cost $\bar{z}_k - c_k = c_B B^{-1}\bar{A}_k - c_k$

   if $\bar{z}_k - c_k \leq 0$ the solution remains optimal,
   if $\bar{z}_k - c_k > 0$ then x_k must enter the basis, proceeding as in the usual simplex algorithm

2. if x_k is a basic variable

   the new set of basic vectors may no longer form a valid basis;
   even if it does, modifying the column of a basic variable changes the basis and therefore B^{-1}, which leads to a modification of the entire simplex table

Example 1

Formulation and the corresponding optimal table

$$\begin{array}{ll} \max & z = 2x_1 + x_2 - x_3 \\ \text{s. a:} & \\ & x_1 + 2x_2 + x_3 \leq 8 \\ & -x_1 + x_2 - 2x_3 \leq 4 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

x_B	x_1	x_2	x_3	x_4	x_5	$\bar{b}$
x_1	1	2	1	1	0	8
x_5	0	3	-1	1	1	12
$z_j - c_j$	0	3	3	2	0	16

Without solving the problem from the beginning, obtain a new optimal solution when the initial data of the problem are changed.

- 3. (a) $\bar{A}_3^t = [1\ 2]$ (b) $\bar{A}_3^t = [-2\ 3]$

Example 2 – ToyCo Example (Taha) – Replacing a Product

New decision: ToyCo considers replacing trains with a new product:

- **police cars**

Characteristics of the new product:

- Unit revenue: \$3
- Processing times (minutes):

	M_1	M_2	M_3
police cars	1	1	2

Question:

- Is it profitable to replace trains with police cars in production?

Method of analysis:

- Treat as a change in column A_1 to $\bar{A}_1$.
- Compute its new **reduced cost**: $\bar{z}_1 - c_1 = c_B B^{-1} \bar{A}_1 - c_1$
- If $\bar{z}_1 - c_1 \leq 0$, it is not worth replacing.
- If $\bar{z}_1 - c_1 > 0$, it should enter the basis.

III – Introduction of a new variable

III – Introduction of a new variable (activity)

Consider a new variable x_{n+1} with cost c_{n+1} and column in the constraint matrix A_{n+1}

without solving the problem we can determine whether it will or will not be advantageous to produce (bring into the basis) x_{n+1}, for that compute

$$z_{n+1} - c_{n+1} = c_B B^{-1} A_{n+1} - c_{n+1}$$

- if $z_{n+1} - c_{n+1} \leq 0$ the solution remains optimal and we will have $x_{n+1} = 0$ in the optimal solution,
- if $z_{n+1} - c_{n+1} > 0$ then x_{n+1} must enter the basis, compute $B^{-1} A_{n+1}$ and proceed as usual with the simplex algorithm

Example 1

Formulation and the corresponding optimal table

$$\begin{array}{ll} \max & z = 2x_1 + x_2 - x_3 \\ \text{s. a:} & \\ & x_1 + 2x_2 + x_3 \leq 8 \\ & -x_1 + x_2 - 2x_3 \leq 4 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

x_B	x_1	x_2	x_3	x_4	x_5	$\bar{b}$
x_1	1	2	1	1	0	8
x_5	0	3	−1	1	1	12
$z_j - c_j$	0	3	3	2	0	16

Without solving the problem from the beginning, obtain a new optimal solution when the initial data of the problem are changed.

- 4. (a) $c_6 = 4$ and $A_6^t = [1\ 2]$ (b) $c_6 = 2$ and $A_6^t = [2\ 3]$

Example 2 – ToyCo Example (Taha) – Introducing a New Product

New decision: ToyCo is considering producing a new product: **Fire trucks** (x_7)
Characteristics of the new product:

- Unit revenue: $4
- Processing times (minutes):

	M_1	M_2	M_3
Fire truck	1	1	2

Question:

- Is it profitable to include fire trucks in production?

Method of analysis:

- Treat x_7 as a **new variable**.
- Compute its **reduced cost**: $\bar{z}_7 - c_7 = c_B B^{-1} A_7 - c_7$
- If $\bar{z}_7 - c_7 \leq 0$, it is not worth adding.
- If $\bar{z}_7 - c_7 > 0$, it should enter the basis.

Example 2 – ToyCo Example (Taha) – Introducing a New Product

New decision: ToyCo is considering producing a new product: **Trailers** (x_7)

Characteristics of trailers:

- Unit revenue: $4
- Processing times (minutes):

	M_1	M_2	M_3
Trailer	1	2	1

Questions:

- Is it worthwhile to produce trailers?
- What are the consequences for the optimal plan and total revenue?

Method of analysis:

- Treat x_5 as a **new variable**.
- Compute its reduced cost: $\bar{z}_7 - c_7 = c_B B^{-1} A_7 - c_7$
- If $\bar{z}_7 - c_7 \leq 0$:
  - The current optimal solution remains optimal.
- If $\bar{z}_7 - c_7 > 0$:
  - The basis must be updated (simplex pivot).

IV – Discrete changes in the RHS vector

IV – Discrete changes in the RHS vector b

the right-hand-side vector is modified from b to $\tilde{b}$

update the last column of the simplex table:

compute $B^{-1}\tilde{b}$ as well as the new value of the objective function $c_B B^{-1}\tilde{b}$

since the previous table was optimal, after

- if primal feasibility is satisfied, $x_B = B^{-1}\tilde{b} \geq 0$, the current basis is optimal and we have a new optimal solution and new optimal objective function value,
- if primal feasibility is violated, proceed with the **dual simplex** algorithm to restore primal feasibility,

Example 1

Formulation and the corresponding optimal table

$$\begin{array}{ll} \max & z = 2x_1 + x_2 - x_3 \\ \text{s. a:} & \\ & x_1 + 2x_2 + x_3 \le 8 \\ & -x_1 + x_2 - 2x_3 \le 4 \\ & x_1, x_2, x_3 \ge 0 \end{array}$$

x_B	x_1	x_2	x_3	x_4	x_5	$\bar{b}$
x_1	1	2	1	1	0	8
x_5	0	3	-1	1	1	12
$z_j - c_j$	0	3	3	2	0	16

Without solving the problem from the beginning, obtain a new optimal solution when the initial data of the problem are changed.

- 5. (a) $\tilde{b}^t = [10 \quad 1]$ (b) $\tilde{b}^t = [3 \quad -12]$

Example 2 – ToyCo Example (Taha) – Increase in Production Capacity

New situation: ToyCo increases its available production time to:

$$M_1 = 600, \quad M_2 = 640, \quad M_3 = 590$$

Questions:

- Will the company still produce only trucks and cars?
- How is the daily revenue affected?

Type of change:

- This is a **change in the RHS vector** b.
- The optimal basis may remain valid.
- If primal feasibility is violated, apply the **dual simplex method**.

Procedure:

- Compute the new basic solution: $x_B = B^{-1}\bar{b}$
- If $x_B \geq 0$:
  - The same basis remains optimal. The values of the optimal solution change.
- Otherwise:
  - Re-optimize using dual simplex.

Example 2 – ToyCo Example (Taha)

New situation: ToyCo increases its available production time to:

$$M_1 = 600, \quad M_2 = 640, \quad M_3 = 590$$

$$\begin{array}{ll} \text{max} & z = 3x_1 + 2x_2 + 5x_3 \\ \text{s.t.} & x_1 + 2x_2 + x_3 \leq 430 \quad (M_1) \\ & 3x_1 + 0x_2 + 2x_3 \leq 460 \quad (M_2) \\ & x_1 + 4x_2 + 0x_3 \leq 420 \quad (M_3) \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

x_B	x_1	x_2	x_3	x_4	x_5	x_6	$\bar{b}$
x_2	$-1/4$	1	0	$1/2$	$-1/4$	0	100
x_3	$3/2$	0	1	0	$1/2$	0	230
x_6	2	0	0	-2	1	1	20
$z_j - c_j$	4	0	0	1	2	0	1350

$$b^T = [430\ 460\ 420] \quad \rightarrow \quad \tilde{b}^T = [600\ 640\ 590]$$

Example 2 – ToyCo Example (Taha) – Major Restructuring

New situation: Due to a major restructuring, the available time becomes:

$$M_1 = 100, \quad M_2 = 600, \quad M_3 = 100$$

Questions:

- Will the company still produce only trucks and cars?
- How is the daily revenue affected?

Type of change:

- This is a **large change in the RHS vector** b.
- The previous optimal basis is likely to become infeasible.
- Re-optimization may be required.

Method:

- Compute the new basic solution: $x_B = B^{-1}\bar{b}$
- If $x_B \geq 0$, the basis remains optimal. New values for the optimal solution.
- Otherwise, apply the **dual simplex method**.

Example 2 – ToyCo Example (Taha)

New situation: ToyCo does a major restructuring, the available production time decreases to:

$$M_1 = 100, \quad M_2 = 600, \quad M_3 = 100$$

$$\begin{array}{ll} \text{max} & z = 3x_1 + 2x_2 + 5x_3 \\ \text{s.t.} & x_1 + 2x_2 + x_3 \leq 430 \quad (M_1) \\ & 3x_1 + 0x_2 + 2x_3 \leq 460 \quad (M_2) \\ & x_1 + 4x_2 + 0x_3 \leq 420 \quad (M_3) \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

x_B	x_1	x_2	x_3	x_4	x_5	x_6	$\bar{b}$
x_2	$-1/4$	1	0	$1/2$	$-1/4$	0	100
x_3	$3/2$	0	1	0	$1/2$	0	230
x_6	2	0	0	-2	1	1	20
$z_j - c_j$	4	0	0	1	2	0	1350

$$b^T = [430 \ 460 \ 420] \quad \rightarrow \quad \tilde{b}^T = [100 \ 600 \ 100]$$

Example 2 – ToyCo Example (Taha) – Post-Optimality Change

New situation:

- Suppose the available time for operation M_2 changes:

$$460 \rightarrow 420$$

Question:

- Do we need to solve the problem from scratch?

Observation:

- The current optimal basis is likely still **dual feasible**
- But it may become **primal infeasible**

Idea:

- Use the **dual simplex method** to restore feasibility efficiently

Dual Simplex Method

Why Dual Simplex?

When to use:

- Initial solution is **dual feasible but primal infeasible**
- After changes in the RHS
- After adding new constraints (re-optimization)

This is the setting for:

$$\boxed{\text{Dual Simplex Method}}$$

- Avoids restarting from scratch
- Uses previous optimal table (when available)

Dual Simplex Method

- Reduced costs satisfy dual feasibility

$$z_j - c_j \geq 0 \quad \Rightarrow \text{dual feasible(maximization primal)}$$

$$z_j - c_j \leq 0 \quad \Rightarrow \text{dual feasible(minimization primal)}$$

- But:

$$x_B = B^{-1}b \ngeq 0 \quad \Rightarrow \text{primal infeasible}$$

Key idea:

- In the **primal simplex** (maximization)
  - Maintain **primal feasibility** ($x_B \geq 0$)
  - Obtain primal optimality (dual feasibility) ($z_j - c_j \geq 0$)
- In the **dual simplex**, we do the opposite:
  - Maintain **dual feasibility** ($z_j - c_j \geq 0$)
  - Obtain (or restore) **primal feasibility** ($x_B \geq 0$)

Dual Feasibility and Optimality

Dual feasibility condition (primal max):

$$z_j - c_j = c_B B^{-1} A_j - c_j \geq 0 \quad \forall j$$

Primal feasibility condition:

$$x_B = B^{-1} b \geq 0$$

Optimality condition:

$$\text{Primal feasible} + \text{Dual feasible} \;\Rightarrow\; \text{Optimal solution}$$

Dual simplex strategy:

- Keep $z_j - c_j \geq 0$
- Eliminate negative components of x_B

Geometric Interpretation

- Primal simplex:
  - Moves along feasible region edges
- Dual simplex:
  - Starts outside feasible region
  - Moves toward feasibility
- Each pivot:
  - Improves feasibility
  - Keeps dual optimal structure

Outline

Step 1: Leaving variable

- Choose the most infeasible basic variable:

$$r = \arg\min_i \{\bar{b}_i\}, \quad \bar{b}_r < 0$$

- This corresponds to the **most violated constraint**

Step 2: Entering variable

- Only consider columns with:

$$\bar{a}_{rj} < 0$$

- Ratio test:

$$\frac{|z_j - c_j|}{|\bar{a}_{rj}|} \quad \text{(choose minimum)}$$

Goal:

- Fix infeasibility without breaking dual feasibility

Why the Ratio Test Works

- After pivoting, reduced costs must remain:

$$z_j - c_j \geq 0$$

- The ratio test ensures:
  - Dual feasibility is preserved
  - No reduced cost becomes negative
- If no $\overline{a}_{rj} < 0$ exists:
  - No way to restore feasibility
  - $\Rightarrow$ Dual unbounded
  - $\Rightarrow$ Primal infeasible

Algorithm Summary (primal: maximization)

Repeat:

1. Compute $\bar{b} = B^{-1}b$
2. If $\bar{b} \geq 0$:

   STOP (optimal solution)

3. Choose leaving variable:

$$r = \arg\min \bar{b}_i$$

4. If $\bar{a}_{rj} \geq 0$ for all j:

   STOP (primal infeasible)

5. Choose entering variable:

$$k = \arg\min \left\{ \frac{|z_j - c_j|}{|\bar{a}_{rj}|} \;:\; \bar{a}_{rj} < 0 \right\}$$

6. Pivot and update basis

Dual Simplex Method for a Minimization Primal Problem

0. Choose an initial dual feasible and basic solution (i.e. such that $z_j - c_j = c_B B^{-1} A_j - c_j \leq 0$ for all j) and let B be the associated basis.
1. Solve the system $B\bar{b} = b$ and let $\bar{b}$ be its unique solution.
   If $\bar{b} \geq 0$, $STOP$, the current basic solution is an optimal solution.
   Otherwise, select the pivot row r such that $\bar{b}_r = \min_{i \in I}\{\bar{b}_i\} < 0$.
   The variable x_r leaves the basis.
2. If $\bar{a}_{rj} \geq 0$, for all j, $STOP$, the dual solution is unbounded and the primal is infeasible.
   Otherwise, the index k of the pivot column is determined by the following ratio
   $$\frac{z_k - c_k}{\bar{a}_{rk}} = \min_{j \in J}\left\{\frac{z_j - c_j}{\bar{a}_{rj}} \; : \bar{a}_{rj} < 0\right\}.$$
   The variable x_k enters the basis.
3. Update the basis B where column A_k replaces column A_{B_r}.
   Update the index sets I_B and I_N.
   Return to step 1.

Example 1

$$
\begin{array}{ll}
\min & z = 4x_1 + 12x_2 + 18x_3 \\
\text{s. t.:} & \\
& x_1 \quad + 3x_3 \geq 3 \\
& \quad 2x_2 + 2x_3 \geq 5 \\
& x_1, x_2, x_3 \geq 0
\end{array}
$$

Example 2

$$
\begin{aligned}
\max \quad & z = -10x_1 - 5x_2 \\
\text{s. t.:} \quad & \\
& -20x_1 - 50x_2 \leq -200 \\
& -50x_1 - 10x_2 \leq -150 \\
& -30x_1 - 30x_2 \leq -210 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

initial basic solution

$$
x = (0, 0, -200, -150, -210) \qquad \text{and} \qquad y = (0, 0, 0, 10, 5)
$$

V – Adding a New Constraint

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1. If the current optimal solution satisfies the new constraint,
   the solution remains optimal for the updated problem.
2. If the current optimal solution does not satisfy the new constraint,
   the solution is no longer optimal for the updated problem
   Let B be the optimal basis before adding the new constraint. The new basis is $\bar{B} = \left[\begin{array}{c|c} B & 0 \\ \hline A_B^{m+1} & \pm 1 \end{array}\right]$ and the corresponding inverse is $\bar{B}^{-1} = \left[\begin{array}{c|c} B^{-1} & 0 \\ \hline \mp A_B^{m+1} B^{-1} & \pm 1 \end{array}\right]$.

2.1 Let $A^{m+1}x \leq b_{m+1}$ be the new constraint and x_{n+1} the corresponding *slack* variable. We can rewrite the constraint as

$$A_B^{m+1}x_B + A_N^{m+1}x_N + x_{n+1} = b_{m+1}.$$

Since the current solution is $x_B = B^{-1}b - B^{-1}Nx_N$, we obtain

$$(A_N^{m+1} - A_B^{m+1}B^{-1}N)x_N + x_{n+1} = b_{m+1} - A_B^{m+1}B^{-1}b.$$

Adding this row to the simplex table with basic variable x_{n+1} yields a basic solution to the new problem.

If $b_{m+1} - A_B^{m+1}B^{-1}b \geq 0$, the solution is optimal.

If $b_{m+1} - A_B^{m+1}B^{-1}b < 0$, the dual simplex algorithm is used to restore primal feasibility.

2.2 Let $A^{m+1}x \geq b_{m+1}$ be the new constraint and x_{n+1} the corresponding *slack* variable. We can rewrite the constraint as

$$A_B^{m+1}x_B + A_N^{m+1}x_N - x_{n+1} = b_{m+1}.$$

Since the tableau solution is $x_B = B^{-1}b - B^{-1}Nx_N$, we obtain

$$(A_B^{m+1}B^{-1}N - A_N^{m+1})x_N + x_{n+1} = A_B^{m+1}B^{-1}b - b_{m+1}.$$

Adding this row to the simplex tableau with basic variable x_{n+1} yields a basic solution to the new problem.

If $A_B^{m+1}B^{-1}b - b_{m+1} \geq 0$, the solution is optimal.

If $A_B^{m+1}B^{-1}b - b_{m+1} < 0$, the dual simplex algorithm is used to restore primal feasibility.

2.3 Let $A^{m+1}x = b_{m+1}$ be the new constraint and x^a_{n+1} the corresponding artificial variable. We can rewrite the constraint as

$$A_B^{m+1}x_B + A_N^{m+1}x_N \pm x^a_{n+1} = b_{m+1},$$

where the artificial variable enters with a $+$ or $-$ sign depending on whether the constraint is violated by excess or by deficiency, respectively.

Example 1

Formulation and the corresponding optimal table

$$\begin{aligned} \max \quad & z = 2x_1 + x_2 - x_3 \\ \text{s. a:} \quad & \\ & x_1 + 2x_2 + x_3 \leq 8 \\ & -x_1 + x_2 - 2x_3 \leq 4 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

x_B	x_1	x_2	x_3	x_4	x_5	$\bar{b}$
x_1	1	2	1	1	0	8
x_5	0	3	-1	1	1	12
$z_j - c_j$	0	3	3	2	0	16

Without solving the problem from the beginning, obtain a new optimal solution when the initial data of the problem are changed.

- 6.
  - (a) $x_1 + x_2 + x_3 \leq 10$
  - (b) $x_2 + x_3 = 10$
  - (c) $x_2 + x_3 \geq 2$
  - (d) $2x_1 - x_2 = 1$