



Lisbon School  
of Economics  
& Management  
Universidade de Lisboa

A decorative graphic at the top of the slide consists of a blue line with circular markers and a green area chart, set against a background of vertical dashed lines.

# STATISTICS I

## Bachelor's degrees in Economics and Finance

### 2<sup>nd</sup> Year/2<sup>nd</sup> Semester

### 2025/2026

# Practical Class Nº 9

**Professor:** Elisabete Fernandes

**E-mail:** efernandes@iseg.ulisboa.pt



<https://doity.com.br/estatistica-aplicada-a-nutricao>

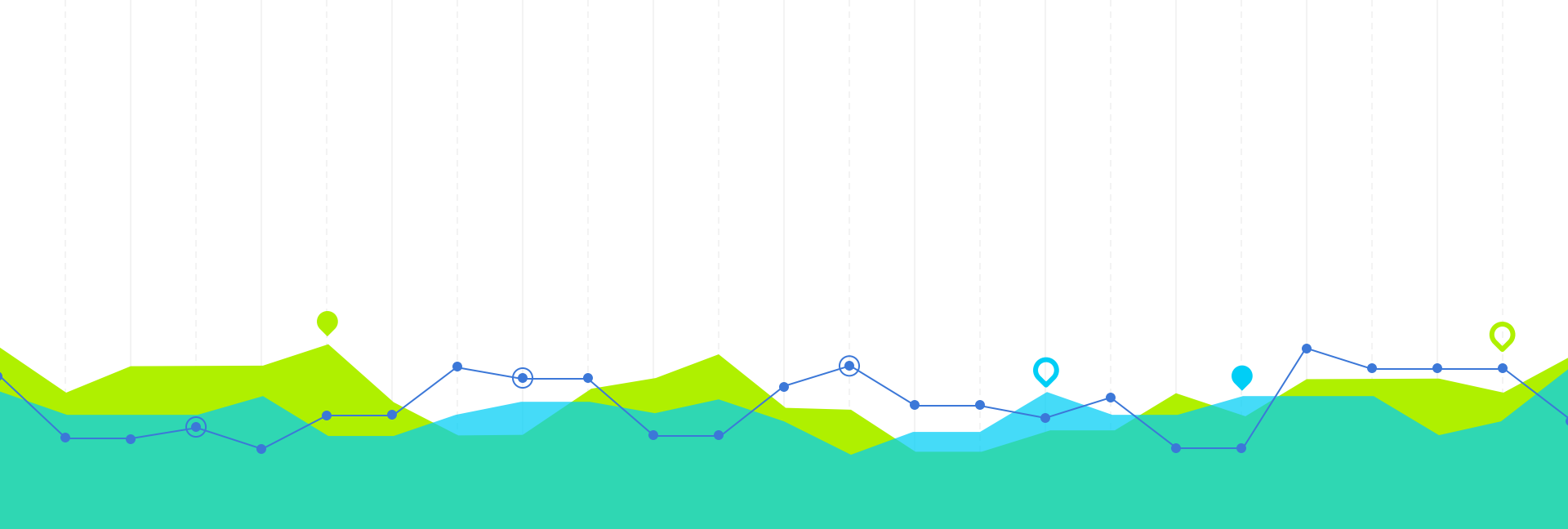


<https://basiccode.com.br/produto/informatica-basica/>

1. Basic Probability Theory.
2. Univariate random variables.
3. Expected Values.
4. Multivariate random variables (random vectors).
5. Expected Values of Functions of Random Vectors.
6. Special Random Variables and Repeated Sampling Distributions.

**Bibliography:**

- Miller & Miller, John E. , Freund's Mathematical Statistics with applications , 8th Edition, Pearson Education, [MM], 2013
- P. Newbold, W. Carlson, B. Thorne, , Statistics for Business and Economics , 8th Edition, Pearson Education, [N], 2012



# Two-Dimensional Continuous Variables: Exercises

Expectations and Parameters for two Dimensional Random Variables

Chapter 5

1

2. Let  $X$  and  $Y$  be two random variables such that

$$f_{X,Y}(x,y) = \begin{cases} e^{-x-y}, & x > 0, y > 0 \\ 0, & \text{otherwise} \end{cases}$$

Compute  $E(XY)$  and  $E(X)$ .



## Exercise 2

$$f_{X,Y}(x,y) = \begin{cases} e^{-x-y}, & x > 0, y > 0 \\ 0, & \text{otherwise} \end{cases}$$

$$E[g(X,Y)] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(x,y) f_{X,Y}(x,y) dx dy$$

$$E[(XY)] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} xy f_{X,Y}(x,y) dx dy = \int_0^{+\infty} \int_0^{+\infty} xy e^{-x-y} dx dy =$$

$$= \int_0^{+\infty} ye^{-y} \int_0^{+\infty} xe^{-x} dx dy = \int_0^{+\infty} ye^{-y} \lim_{a \rightarrow +\infty} [-e^{-x}(x+1)]_0^a dy =$$

$$= \int_0^{+\infty} ye^{-y} \lim_{a \rightarrow +\infty} (-e^{-a}(a+1) - (-e^0(0+1))) dy =$$

$$= \int_0^{+\infty} ye^{-y} \left( \lim_{a \rightarrow +\infty} \underbrace{-ae^{-a}}_{\frac{a}{e^a} \rightarrow 0} - \lim_{a \rightarrow +\infty} \underbrace{e^{-a}}_{\frac{1}{e^a} \rightarrow 0} + 1 \right) dy$$

$$= \int_0^{+\infty} ye^{-y} dy = \lim_{b \rightarrow +\infty} [-e^{-y}(y+1)]_0^b =$$

$$= \lim_{b \rightarrow +\infty} (-e^{-b}(b+1) - (-e^0(0+1))) = \lim_{b \rightarrow +\infty} \underbrace{-be^{-b}}_0 - \lim_{b \rightarrow +\infty} \underbrace{e^{-b}}_0 + 1 =$$

$$= 1$$

## Exercise 2

$$\int (u v') = u v - \int (v u')$$

$$\begin{array}{l} u = x \\ v' = e^{-x} \end{array} \quad \begin{array}{l} u' = 1 \\ v = \int v' = -e^{-x} \end{array}$$

$$\begin{aligned} \int x e^{-x} &= x(-e^{-x}) - \int -e^{-x}(1) = \\ &= -x e^{-x} + \int e^{-x} = -x e^{-x} - e^{-x} = \\ &= -e^{-x}(1+x) \end{aligned}$$

## Exercise 2

$$\begin{aligned} E(X) &= \int_0^{+\infty} \int_0^{+\infty} x f_{X,Y}(x,y) dy dx = \int_0^{+\infty} \int_0^{+\infty} x e^{-x-y} dy dx = \\ &= \int_0^{+\infty} x e^{-x} \int_0^{+\infty} e^{-y} dy dx = \int_0^{+\infty} x e^{-x} \lim_{a \rightarrow +\infty} [-e^{-y}]_0^a dx = \\ &= \int_0^{+\infty} x e^{-x} \lim_{a \rightarrow +\infty} (-e^{-a} - (-e^0)) dx = \int_0^{+\infty} x e^{-x} (\underbrace{\lim_{a \rightarrow +\infty} (-e^{-a})}_0 + 1) dx = \\ &= \int_0^{+\infty} x e^{-x} dx = \lim_{b \rightarrow +\infty} [-e^{-x}(1+x)]_0^b = \\ &= \lim_{b \rightarrow +\infty} (-e^{-b}(1+b) - \underbrace{(-e^0(1+0))}_1) = \\ &= \underbrace{\lim_{b \rightarrow +\infty} (-e^{-b})}_0 - \underbrace{\lim_{b \rightarrow +\infty} (b e^{-b})}_0 + 1 = 1 \end{aligned}$$

4. If the probability density of  $X$  is given by

$$f(x) = \begin{cases} 1+x & , \text{for } -1 < x \leq 0 \\ 1-x & , \text{for } 0 < x < 1 \\ 0 & , \text{elsewhere} \end{cases}$$

and  $U = X$  and  $V = X^2$ , show that

(a)  $cov(U, V) = 0$ ;

(b)  $U$  and  $V$  are dependent.



## Exercise 4 a)

a)

$$\begin{aligned} E(X) &= \int_{-\infty}^{+\infty} x f_X(x) dx = \int_{-1}^0 x(1+x) dx + \int_0^1 x(1-x) dx = \\ &= \int_{-1}^0 x dx + \int_{-1}^0 x^2 dx + \int_0^1 x dx + \int_0^1 -x^2 dx = \\ &= \left[ \frac{x^2}{2} \right]_{-1}^0 + \left[ \frac{x^3}{3} \right]_{-1}^0 + \left[ \frac{x^2}{2} \right]_0^1 - \left[ \frac{x^3}{3} \right]_0^1 = \\ &= 0 - \frac{(-1)^2}{2} + 0 - \frac{(-1)^3}{3} + \frac{1}{2} - 0 - \frac{1}{3} - 0 = \\ &= -\frac{1}{2} + \frac{1}{3} + \frac{1}{2} - \frac{1}{3} = 0 \end{aligned}$$

$$\begin{aligned} E(X^3) &= \int_{-\infty}^{+\infty} x^3 f_X(x) dx = \int_{-1}^0 x^3(1+x) dx + \int_0^1 x^3(1-x) dx = \\ &= \int_{-1}^0 x^3 dx + \int_{-1}^0 x^4 dx + \int_0^1 x^3 dx + \int_0^1 -x^4 dx = \\ &= \left[ \frac{x^4}{4} \right]_{-1}^0 + \left[ \frac{x^5}{5} \right]_{-1}^0 + \left[ \frac{x^4}{4} \right]_0^1 - \left[ \frac{x^5}{5} \right]_0^1 = \\ &= 0 - \frac{(-1)^4}{4} + 0 - \frac{(-1)^5}{5} + \frac{1}{4} - 0 - \frac{1}{5} - 0 = \\ &= -\frac{1}{4} + \frac{1}{5} + \frac{1}{4} - \frac{1}{5} = 0 \end{aligned}$$

$$\begin{aligned} \text{cov}(U, V) &= \text{cov}(X, X^2) = E(X X^2) - E(X)E(X^2) = \\ &= \underbrace{E(X^3)}_0 - \underbrace{E(X)E(X^2)}_0 = 0, \quad Q \in \mathbb{D} \end{aligned}$$

## Exercise 4 b)

We need to show that:  $\exists (u, v) \in \mathbb{R}^2: f_{u,v}(u, v) \neq f_u(u) f_v(v)$

Since  $U = X$  we have:  $f_u(u) = f_x(u) = \begin{cases} 1+u & (-1 < u < 0) \\ 1-u & (0 < u < 1) \\ 0 & (\text{elsewhere}) \end{cases}$

Now let's find  $f_v(v)$  ( $0 < v < 1$ ):

$$\begin{aligned} F_v(v) &= P(V \leq v) = P(X^2 \leq v) = P(-\sqrt{v} \leq X \leq \sqrt{v}) = \\ &= F_x(\sqrt{v}) - F_x(-\sqrt{v}) \quad (0 < v < 1) \end{aligned}$$

$$\begin{aligned} f_v(v) &= F'_x(\sqrt{v}) \frac{d}{dv}(\sqrt{v}) - F'_x(-\sqrt{v}) \frac{d}{dv}(-\sqrt{v}) = \\ &= f_x(\sqrt{v}) \frac{1}{2\sqrt{v}} - f_x(-\sqrt{v}) \frac{-1}{2\sqrt{v}} = \\ &= \frac{f_x(\sqrt{v})}{2\sqrt{v}} + \frac{f_x(-\sqrt{v})}{2\sqrt{v}} = \frac{1-\sqrt{v}}{2\sqrt{v}} + \frac{1+\sqrt{v}}{2\sqrt{v}} = \frac{2-2\sqrt{v}}{2\sqrt{v}} = \frac{1-\sqrt{v}}{\sqrt{v}} \quad (0 < v < 1) \end{aligned}$$

## Exercise 4 b)

$$\text{So, } f_v(r) = \begin{cases} \frac{1-\sqrt{r}}{\sqrt{r}} & (0 < r < 1) \\ 0 & (\text{otherwise}) \end{cases}$$

Now we only have to find  $f_{u,v}(u, r)$ .

Since  $U = X$  and  $V = X^2$  then  $V = U^2$  and  $f_{u,v}(u, r) > 0$  only if  $r = u^2$  (and  $-1 < u < 1$ ).

from this,  $V$  and  $U$  are obviously dependent!

$$\text{So, } f_{u,v}(u, r) = \begin{cases} f_x(u) & (\text{if } r = u^2 \wedge -1 < u < 1) \\ 0 & (\text{otherwise}) \end{cases} = \begin{cases} 1+u & (-1 < u < 0, r = u^2) \\ 1-u & (0 < u < 1, r = u^2) \\ 0 & (\text{elsewhere}) \end{cases}$$

Conclusion:

$$f_{u,r}(u, r) \neq f_u(u) f_r(r) = \begin{cases} (1+u)(1-\sqrt{r})/\sqrt{r} & (-1 < u < 0, 0 < r < 1) \\ (1-u)(1-\sqrt{r})/\sqrt{r} & (0 < u < 1, 0 < r < 1) \\ 0 & (\text{elsewhere}) \end{cases}$$

Therefore  $X \not\perp Y$ , QED

6. If the joint probability density of  $X$  and  $Y$  is given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{3}(y+x) & , \text{for } 0 < x \leq 1, 0 < y < 2 \\ 0 & , \text{elsewhere} \end{cases}$$

Compute  $Var(X|Y = y)$ .



## Exercise 6

$$\begin{aligned}
 \text{Var}(X|Y=y) &= E(X^2|Y=y) - E(X|Y=y)^2 = \frac{3+4y}{6+12y} - \left(\frac{2+3y}{3+6y}\right)^2 = \\
 &= \frac{3+4y}{2(3+6y)} - \frac{4+9y^2+12y}{(3+6y)^2} = \\
 &= \frac{(3+4y)(3+6y) - 2(4+9y^2+12y)}{2(3+6y)^2} = \\
 &= \frac{9+18y+12y+24y^2-8-18y^2-24y}{2(9+36y^2+36y)} = \frac{6y^2+6y+1}{18+72y^2+72y} = \\
 &= \frac{6y^2+6y+1}{18(1+4y^2+4y)} = \frac{6y^2+4y+(1+2y)}{18(1+2y)^2} = \frac{1}{18} \left( \frac{6y^2+4y}{(1+2y)^2} + \frac{1+2y}{(1+2y)^2} \right) = \\
 &= \frac{1}{36} \left( 2 \frac{6y^2+4y}{(1+2y)^2} + \frac{2}{1+2y} \right) = \frac{1}{36} \left( \frac{12y^2+8y+2(1+2y)}{(1+2y)^2} \right) = \\
 &= \frac{1}{36} \left( \frac{12y^2+8y+2+4y}{(1+2y)^2} \right) = \frac{1}{36} \left( \frac{12y^2+12y+2}{(1+2y)^2} \right) = \\
 &= \frac{1}{36} \left( \frac{12y^2+12y+3}{(1+2y)^2} - \frac{1}{(1+2y)^2} \right) = \frac{1}{36} \left( 3 \frac{(4y^2+4y+1)}{(1+2y)^2} - \frac{1}{(2y+1)^2} \right) \\
 &= \frac{1}{36} \left( 3 \frac{(1+2y)^2}{(1+2y)^2} - \frac{1}{(2y+1)^2} \right) = \frac{1}{36} \left( 3 - \frac{1}{(2y+1)^2} \right) \quad (0 < y < 2) \rightarrow \text{same as solutions}
 \end{aligned}$$

This was enough

## Exercise 6

Auxiliary calculations:

$$\begin{aligned}f_Y(y) &= \int_{-\infty}^{+\infty} f_{X,Y}(x,y) dx = \int_0^1 \frac{1}{3}(x+y) dx = \\ &= \frac{1}{3} \left[ \frac{x^2}{2} + xy \right]_{x=0}^{x=1} = \frac{1}{3} \left( \frac{1}{2} + y \right) \quad (0 < y < 2)\end{aligned}$$

$$\begin{aligned}E(X|Y=y) &= \int_{-\infty}^{+\infty} x f_{X|Y=y}(x) dx = \int_{-\infty}^{+\infty} x \frac{f_{X,Y}(x,y)}{f_Y(y)} dx \\ &= \int_0^1 x \frac{\frac{1}{3}(y+x)}{\frac{1}{3}(\frac{1}{2}+y)} dx = \int_0^1 x \frac{y+x}{y+\frac{1}{2}} dx = \int_0^1 \frac{xy+x^2}{y+\frac{1}{2}} dx = \\ &= \frac{1}{y+\frac{1}{2}} \int_0^1 xy+x^2 dx = \frac{1}{y+\frac{1}{2}} \left[ \frac{x^2 y}{2} + \frac{x^3}{3} \right]_{x=0}^{x=1} \\ &= \frac{\frac{y}{2} + \frac{1}{3}}{\frac{1}{2} + y} = \frac{\frac{2}{3} + y}{1+2y} = \frac{\frac{6}{3} + 3y}{3+6y} = \frac{2+3y}{3+6y} \quad (0 < y < 2) \rightarrow \text{same as solutions}\end{aligned}$$

This was enough

## Exercise 6

$$\begin{aligned} P(X^2|Y=y) &= \int_{-\infty}^{+\infty} x^2 f_{X|Y=y}(x) dx = \int_{-\infty}^{+\infty} x^2 \frac{f_{X,Y}(x,y)}{f_Y(y)} dx \\ &= \int_0^1 x^2 \frac{\frac{1}{3}(y+x)}{\frac{1}{3}(\frac{1}{2}+y)} dx = \int_0^1 x^2 \frac{y+x}{y+\frac{1}{2}} dx = \int_0^1 \frac{x^2 y + x^3}{y+\frac{1}{2}} dx = \\ &= \frac{1}{y+\frac{1}{2}} \int_0^1 x^2 y + x^3 dx = \frac{1}{y+\frac{1}{2}} \left[ \frac{x^3 y}{3} + \frac{x^4}{4} \right]_{x=0}^{x=1} \\ &= \frac{\frac{y}{3} + \frac{1}{4}}{\frac{1}{2} + y} = \frac{y + \frac{3}{4}}{\frac{3}{2} + 3y} = \frac{3 + 4y}{\frac{12}{2} + 12y} = \frac{3 + 4y}{6 + 12y} \quad (0 < y < 2) \rightarrow \text{same as solutions} \end{aligned}$$

This was enough

8. Suppose that  $E(Y|X = x) = 9 + x$ . If the  $E(X) = 11$  and  $Var(X) = 290$ , what is  $cov(Y, X)$ ?



## Exercise 8

Suppose that  $E(Y|X = x) = 9 + x$ . If the  $E(X) = 11$  and  $Var(X) = 290$ , what is  $cov(Y, X)$ ?

*Law of iterated expectations*

$$E(Y) = E(E(Y|X)) = E(9 + X) = 9 + E(X) = 9 + 11 = 20$$

$$E(X^2) = Var(X) + E(X)^2 = 290 + 11^2 = 411$$

$$\begin{aligned} E(\underbrace{XY}_{g(X,Y)}) &= E(E(XY|X)) = E(X E(Y|X)) = E(X(9+X)) = \\ &= E(9X + X^2) = 9E(X) + E(X^2) = 9 \times 11 + 411 = 510 \end{aligned}$$

$$\begin{aligned} cov(X, Y) &= E(XY) - E(X)E(Y) = \\ &= 510 - 11 \times 20 = 510 - 220 = 290 \end{aligned}$$

## Exercise 8

Note (Law of iterated expectations):

If  $X$  and  $Y$  are random variables then:

$$E(X) = E(E(X|Y));$$

$$E(Y) = E(E(Y|X))$$

If  $Z = g(X, Y)$  then:

$$E(g(X, Y)) = E(Z) = E(E(Z|X)) = E(E(g(X, Y)|X)) =$$

$$E(g(X, Y)) = E(Z) = E(E(Z|Y)) = E(E(g(X, Y)|Y)) =$$

9. Let  $X$  and  $Y$  be two continuous random variables such that the conditional density function of  $X$  given  $Y = y$  is

$$f_{X|Y=y}(x) = \begin{cases} \frac{1}{2-y}, & y < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

and the marginal density of  $Y$  is given by

$$f_Y(y) = \begin{cases} 1 - y/2, & 0 < y < 2 \\ 0, & \text{otherwise} \end{cases}.$$

Compute the expected value of  $X$ .



## Exercise 9

Let  $X$  and  $Y$  be two continuous random variables such that the conditional density function of  $X$  given  $Y = y$  is

$$f_{X|Y=y}(x) = \begin{cases} \frac{1}{2-y}, & y < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

and the marginal density of  $Y$  is given by

$$f_Y(y) = \begin{cases} 1 - y/2, & 0 < y < 2 \\ 0, & \text{otherwise} \end{cases}$$

Compute the expected value of  $X$ .

$$\begin{aligned} E(X) &= \int_0^2 \int_Y^2 x f_{X,Y}(x,y) dx dy = \int_0^2 \int_Y^2 \frac{x}{2} dx dy = \\ &= \frac{1}{2} \int_0^2 \left[ \frac{x^2}{2} \right]_{x=Y}^{x=2} dy = \frac{1}{2} \int_0^2 \left( 2 - \frac{y^2}{2} \right) dy = \\ &= \frac{1}{2} \left[ 2y - \frac{y^3}{6} \right]_{y=0}^{y=2} = \frac{1}{2} \left( 4 - \frac{8}{6} \right) = \frac{1}{2} \left( 4 - \frac{4}{3} \right) = \frac{1}{2} \left( \frac{12}{3} - \frac{4}{3} \right) = \\ &= \frac{8}{6} = \frac{4}{3} \end{aligned}$$

*see auxiliary calculation*

## Exercise 9

Auxiliary calculation: (to find  $f_{X,Y}(x,y)$ ):

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \quad (=) \quad \frac{1}{2-y} = \frac{f_{X,Y}(x,y)}{1-\frac{y}{2}} \quad (=)$$

$$(\Rightarrow) \quad \frac{1 - \frac{y}{2}}{2-y} = f_{X,Y}(x,y) \quad (=)$$

$$(\Rightarrow) \quad f_{X,Y}(x,y) = \frac{2-y}{2(2-y)} = \frac{1}{2} \quad (y < x < 2)$$

## Exercise 9

Alternative resolution:

**Solution:** Firstly, we can compute

$$E(X|Y = y) = \int_{-\infty}^{+\infty} x f_{X|Y=y}(x) dx = \int_y^2 \frac{x}{2-y} dx = 1 + y/2.$$

Secondly, we can use the tower property

$$E(X) = E(E(X|Y)) = E(1 + Y/2) = 1 + E(Y)/2.$$

By definition,

$$E(Y) = \int_{-\infty}^{+\infty} y f_Y(y) dy = \int_0^2 y(1 - y/2) dy = 2/3.$$

Therefore  $E(X) = 1 + 1/3 = 4/3$ .

11. A company engaged in the trade of various items, whose sales have random behavior. The monthly sales of items  $A$  and  $B$ , expressed in monetary units, constitute a random vector  $(X, Y)$  with joint probability density function given by:

$$f_{X,Y}(x, y) = \frac{1}{2}, \quad 0 < x < 2, 0 < y < x.$$

- (a) Compute the means and variances of  $X$  and  $Y$ .
- (b) Analyze the independence of the two random variables and compute the correlation coefficient.
- (c) Find the  $E(Y|X = 1)$ .
- (d) Compute the mean and variance of total sales of the two articles.



## Exercise 11 a)

$$f_{X,Y}(x,y) = \frac{1}{2}, \quad 0 < x < 2, \quad 0 < y < x.$$

↳ given  $y$  we have  $y < x < 2$

a)

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y) dy = \int_0^x \frac{1}{2} dy = \frac{1}{2} [y]_{y=0}^{y=x} = \frac{x}{2} \quad (0 < x < 2)$$

$$\begin{aligned} E(X) &= \int_{-\infty}^{+\infty} x f_X(x) dx = \int_0^2 x \frac{x}{2} dx = \frac{1}{2} \int_0^2 x^2 dx = \frac{1}{2} \left[ \frac{x^3}{3} \right]_{x=0}^{x=2} = \\ &= \frac{1}{6} (2^3 - 0) = \frac{8}{6} = \frac{4}{3} \end{aligned}$$

$$\begin{aligned} E(X^2) &= \int_{-\infty}^{+\infty} x^2 f_X(x) dx = \int_0^2 x^2 \frac{x}{2} dx = \frac{1}{2} \int_0^2 x^3 dx = \frac{1}{2} \left[ \frac{x^4}{4} \right]_{x=0}^{x=2} = \\ &= \frac{1}{8} (2^4 - 0) = \frac{16}{8} = 2 \end{aligned}$$

## Exercise 11 a)

$$\text{Var}(X) = E(X^2) - E(X)^2 = 2 - \left(\frac{4}{3}\right)^2 = 2 - \frac{16}{9} = \frac{18}{9} - \frac{16}{9} = \frac{2}{9}$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y) dx = \int_y^2 \frac{1}{2} dx = \frac{1}{2} [x]_{x=y}^{x=2} = \frac{2-y}{2} = 1 - \frac{y}{2} \quad (0 < y < 2)$$

$$\begin{aligned} E(Y) &= \int_{-\infty}^{+\infty} y f_Y(y) dy = \int_0^2 y \left(1 - \frac{y}{2}\right) dy = \int_0^2 y - \frac{y^2}{2} dy = \left[ \frac{y^2}{2} - \frac{y^3}{6} \right]_{y=0}^{y=2} = \\ &= \frac{4}{2} - \frac{8}{6} = 2 - \frac{4}{3} = \frac{6}{3} - \frac{4}{3} = \frac{2}{3} \end{aligned}$$

$$\begin{aligned} E(Y^2) &= \int_{-\infty}^{+\infty} y^2 f_Y(y) dy = \int_0^2 y^2 \left(1 - \frac{y}{2}\right) dy = \int_0^2 y^2 - \frac{y^3}{2} dy = \left[ \frac{y^3}{3} - \frac{y^4}{8} \right]_{y=0}^{y=2} = \\ &= \frac{8}{3} - \frac{16}{8} = \frac{64}{24} - \frac{48}{24} = \frac{16}{24} = \frac{2}{3} \end{aligned}$$

## Exercise 11 a)

$$\text{Var}(Y) = E(Y^2) - E(Y)^2 = \frac{2}{3} - \left(\frac{2}{3}\right)^2 = \frac{2}{3} - \frac{4}{9} = \frac{6}{9} - \frac{4}{9} = \frac{2}{9}$$

## Exercise 11 a)

Alternative solution:

$$\begin{aligned} E(X) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x f_{X,Y}(x,y) dy dx = \int_0^2 \int_0^x \frac{x}{2} dy dx = \\ &= \frac{1}{2} \int_0^2 x [y]_{y=0}^{y=x} dx = \frac{1}{2} \int_0^2 x(x-0) dx = \\ &= \frac{1}{2} \int_0^2 x^2 dx = \frac{1}{2} \left[ \frac{x^3}{3} \right]_{x=0}^{x=2} = \frac{1}{6} (8-0) = \frac{8}{6} = \frac{4}{3} \end{aligned}$$

$$\begin{aligned} E(X^2) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x^2 f_{X,Y}(x,y) dy dx = \int_0^2 \int_0^x \frac{x^2}{2} dy dx = \\ &= \frac{1}{2} \int_0^2 x^2 [y]_{y=0}^{y=x} dx = \frac{1}{2} \int_0^2 x^2(x-0) dx = \\ &= \frac{1}{2} \int_0^2 x^3 dx = \frac{1}{2} \left[ \frac{x^4}{4} \right]_{x=0}^{x=2} = \frac{1}{8} (16-0) = 2 \end{aligned}$$

$$\text{Var}(X) = E(X^2) - E(X)^2 = 2 - \left(\frac{4}{3}\right)^2 = 2 - \frac{16}{9} = \frac{18}{9} - \frac{16}{9} = \frac{2}{9}$$

## Exercise 11 a)

$$\begin{aligned} E(Y) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} y f_{X,Y}(x,y) dy dx = \int_0^2 \int_0^x \frac{y}{2} dy dx = \\ &= \frac{1}{2} \int_0^2 \left[ \frac{y^2}{2} \right]_{y=0}^{y=x} dx = \frac{1}{4} \int_0^2 x^2 - 0 dx = \\ &= \frac{1}{4} \int_0^2 x^2 dx = \frac{1}{4} \left[ \frac{x^3}{3} \right]_{x=0}^{x=2} = \frac{1}{12} (8 - 0) = \frac{8}{12} = \frac{2}{3} \end{aligned}$$

$$\begin{aligned} E(Y^2) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} y^2 f_{X,Y}(x,y) dy dx = \int_0^2 \int_0^x \frac{y^2}{2} dy dx = \\ &= \frac{1}{2} \int_0^2 \left[ \frac{y^3}{3} \right]_{y=0}^{y=x} dx = \frac{1}{6} \int_0^2 x^3 - 0 dx = \\ &= \frac{1}{6} \int_0^2 x^3 dx = \frac{1}{6} \left[ \frac{x^4}{4} \right]_{x=0}^{x=2} = \frac{1}{24} (16 - 0) = \frac{16}{24} = \frac{2}{3} \end{aligned}$$

$$\text{Var}(Y) = E(Y^2) - E(Y)^2 = \frac{2}{3} - \left(\frac{2}{3}\right)^2 = \frac{2}{3} - \frac{4}{9} = \frac{6}{9} - \frac{4}{9} = \frac{2}{9}$$

## Exercise 11 b)

$$f_x(x) f_y(y) = \frac{x}{2} \left(1 - \frac{y}{2}\right) \neq f_{x,y}(x,y) = \frac{1}{2} \quad (0 < x < 2, 0 < y < x)$$

Therefore  $X \neq Y$ .

Alternative solution:

$$\text{for } 0 < x < 2 \wedge x < y < 2 : f_{x,y}(x,y) = 0 \neq f_x(x) f_y(y) = \frac{x}{2} \left(1 - \frac{y}{2}\right) > 0$$

$$\rho_{x,y} = \frac{\text{cov}(X,Y)}{\sigma_x \sigma_y} = \frac{E(XY) - E(X)E(Y)}{\sqrt{\text{Var}(X) \text{Var}(Y)}} = \frac{1 - \frac{4}{3} \cdot \frac{8}{3}}{\sqrt{\frac{2}{9} \times \frac{2}{9}}} = \frac{1 - \frac{8}{9}}{\frac{2}{9}} = \frac{\frac{1}{9}}{\frac{2}{9}} = \frac{1}{2}$$

$$E(XY) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} xy f_{x,y}(x,y) dy dx = \int_0^2 \int_0^x \frac{xy}{2} dy dx =$$

$$= \frac{1}{2} \int_0^2 x \int_0^x y dy dx = \frac{1}{2} \int_0^2 \frac{x}{2} [y^2]_{y=0}^{y=x} dx = \frac{1}{4} \int_0^2 x(x^2 - 0) dx$$

$$= \frac{1}{4} \int_0^2 x^3 dx = \frac{1}{4} \left[ \frac{x^4}{4} \right]_{x=0}^{x=2} = \frac{1}{16} (16 - 0) = 1$$

## Exercise 11 c)

$$\begin{aligned} E(Y | X=1) &= \int_{-\infty}^{+\infty} y f_{Y|X=1}(y) dy = \int_0^1 y \frac{f_{X,Y}(1,y)}{f_X(1)} dy = \\ &= \int_0^1 y \frac{\frac{1}{2}}{\frac{1}{2}} dy = \int_0^1 y dy = \left[ \frac{y^2}{2} \right]_{y=0}^{y=1} = \frac{1}{2} - 0 = \frac{1}{2} \end{aligned}$$

## Exercise 11 d)

d)

$X + Y \equiv$  Total sales

$$E(X+Y) = E(X) + E(Y) = \frac{4}{3} + \frac{2}{3} = \frac{6}{3} = 2$$

$$\begin{aligned} \text{Var}(X+Y) &= \text{Var}(X) + \text{Var}(Y) + 2 \text{cov}(X, Y) = \\ &= \frac{2}{9} + \frac{2}{9} + 2 \times \frac{1}{9} = \frac{6}{9} = \frac{2}{3} \end{aligned}$$

# Thanks!

## Questions?

