Final Exam 2025-2026

Solutions

December 22, 2025

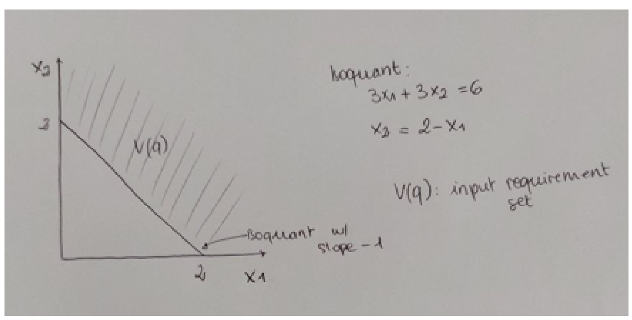
1 Production (3 points)

1.1 Returns to scale

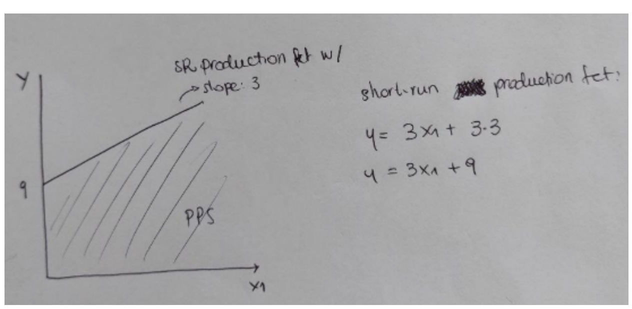
This production function exhibits constant returns to scale since scaling both inputs by any factor $t > 0$ yields

$$f(tx) = \beta(tx_1) + \gamma(tx_2) = t(\beta x_1 + \gamma x_2) = t\, f(x).$$

1.2 Input requirement set sketch



1.3 Short-run production possibilities set sketch

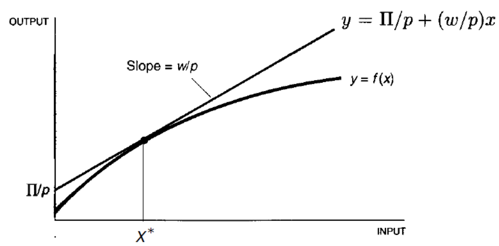


2 Profit and costs (4 points)

2.1 Profit Maximization A profit-maximizing firms that produces 1 output with $n = 1$ input, maximizes the following:

$$\pi(p, w) = py - wx.$$

The problem can be represented graphicall as follows:



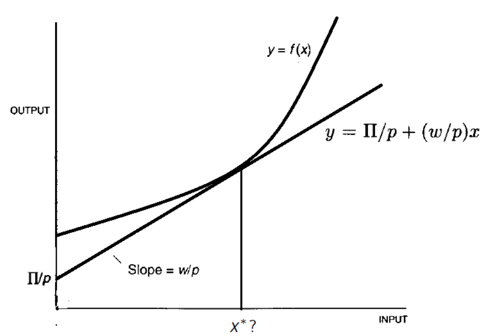
The optimal choice $x^\star$ correspond to a point on the production function $y = f(x)$ with maximal profit. This is a point where the intercept (i.e., $\frac{\Pi}{p}$) of the isoprofit line (i.e., $L(\Pi) = \{(y, x) : y = \left(\frac{\Pi}{p}\right) + \left(\frac{w}{p}\right)x\}$) is maximal. This point x^* is characterized by the slopes of the profit function (i.e., $\partial f(x^*)/\partial x$) and of the isoprofit line (i.e., w/p) being equal, which is the optimality condition:

$$\frac{\partial f(x^*)}{\partial x} = \frac{w}{p}.$$

In this two dimensional case, the **second-order condition** (SOC) is:

$$\frac{\partial^2 f(x)}{\partial x^2} \leq 0.$$

Absent the SOC, the profit maximization is without meaning, as it would lead to the optimal $x^\star$ being a minimum:



2.2 Second-order condition

For $f(x) = x^\alpha$, profits are

$$\pi(x) = px^\alpha - wx.$$

FOC (interior) is

$$p\alpha x^{\alpha-1} - w = 0.$$

SOC for a maximum is

$$p\alpha(\alpha - 1)x^{\alpha-2} \leq 0.$$

Since $p > 0$, $\alpha > 0$, and $x^{\alpha-2} > 0$ for $x > 0$, the sign of the SOC is given by the sign of $(\alpha - 1)$. It follows that $\pi''(x) \leq 0$ requires $\alpha - 1 \leq 0$, i.e. $\alpha \leq 1$.

2.3 Conditional Factor demand

The short-run cost-minimization problem is

$$\min_{x_1} w_1x_1 + w_2x_2 \quad \text{s.t.} \quad x_1^\alpha k^{1-\alpha} = y.$$

Solving for x_1 gives the short-run conditional factor demand:

$$x_1(w, y, x_2 = k) = \left(\frac{y}{k^{1-\alpha}}\right)^{1/\alpha}$$

With one variable input (x_2 is fixed) and a binding requirement, x_1 is pinned down by technology and output.

2.4 Cost functions

Short-run cost function:

$$c(w, y, x_2 = k) = w_1 \left(\frac{y}{k^{1-\alpha}}\right)^{1/\alpha} + w_2k$$

Short-run average variable cost:

$$SAVC(w, y, k) = \frac{c(w, y, k)}{y} = w\,\frac{\left(\frac{y}{k^{1-\alpha}}\right)^{1/\alpha}}{y} = w\,y^{\frac{1}{\alpha}-1}\,k^{-\frac{1-\alpha}{\alpha}}$$

Short-run average fixed cost:

$$SAFC(w, y, k) = \frac{w_2k}{y}$$

Short-run marginal cost:

$$MC(w, y, k) = \frac{\partial c(w, y, k)}{\partial y} = w \cdot \frac{1}{\alpha}\left(\frac{1}{k^{1-\alpha}}\right)^{1/\alpha} y^{\frac{1}{\alpha}-1} = \frac{w}{\alpha}\,\left(\frac{y}{k}\right)^{\frac{1-\alpha}{\alpha}}$$

3 Consumer choice (3 points)

3.1 Marshallian demand

Lagrangian:

$$\mathcal{L} = x_1^{\alpha} x_2^{1-\alpha} - \lambda\big(p_1 x_1 + p_2 x_2 - m\big).$$

FOCs:

$$\alpha x_1^{\alpha-1} x_2^{1-\alpha} = \lambda p_1, \qquad (1-\alpha) x_1^{\alpha} x_2^{-\alpha} = \lambda p_2, \qquad p_1 x_1 + p_2 x_2 = m.$$

From the first two FOCs:

$$\frac{\alpha}{1-\alpha} \cdot \frac{x_2}{x_1} = \frac{p_1}{p_2} \quad \Rightarrow \quad \frac{x_2}{x_1} = \frac{1-\alpha}{\alpha} \cdot \frac{p_1}{p_2}.$$

Use the budget constraint to solve for x_1 and x_2:

$$x_1(p, m) = \frac{\alpha m}{p_1},$$

3.2 Hicksian demand

$$\min_{x_1, x_2} \; p_1 x_1 + p_2 x_2 \quad \text{s.t.} \quad x_1^{\alpha} x_2^{1-\alpha} = u$$

Lagrangian:

$$\mathcal{L} = p_1 x_1 + p_2 x_2 - \lambda\big(x_1^{\alpha} x_2^{1-\alpha} - u\big).$$

FOCs:

$$p_1 = \lambda \alpha x_1^{\alpha-1} x_2^{1-\alpha}, \qquad p_2 = \lambda (1-\alpha) x_1^{\alpha} x_2^{-\alpha}, \qquad x_1^{\alpha} x_2^{1-\alpha} = u$$

Taking ratios:

$$\frac{p_1}{p_2} = \frac{\alpha}{1-\alpha} \frac{x_2}{x_1} \quad \Rightarrow \quad x_2 = \frac{1-\alpha}{\alpha} \frac{p_1}{p_2} x_1.$$

Substitute into the constraint:

$$x_1^{\alpha} \left(\frac{1-\alpha}{\alpha} \frac{p_1}{p_2} x_1 \right)^{1-\alpha} = u \quad \Longleftrightarrow \quad h_1(p, u) = u \left(\frac{\alpha}{1-\alpha} \frac{p_2}{p_1} \right)^{1-\alpha},$$

3.3 Duality

Duality means that, under the standard assumptions for utility functions (completeness, transitivity, and LNS), EMP and UMP solutions will lead to the same optimal point $\mathbf{x}^*$. In other words, at the optimum, the two problems are equivalent as the UMP chooses the best affordable bundle, given a fixed m and EMP chooses the cheapest bundle that achieves a target utility u.

3.4 Slutsky, Marshallian and Hicksian

The Slutsky equation decomposes the Marshallian price effect into a substitution effect and an income effect:

$$\frac{\partial x_1(p,m)}{\partial p_1} = \frac{\partial h_1(p,u)}{\partial p_1} - \frac{\partial x_1(p,m)}{\partial m} x_1(p,m).$$

The Hicksian demand captures only the substitution effect, since utility is held constant. The Marshallian demand captures both the substitution effect and the income effect, since income is held constant.

The Marshallian and Hicksian demands coincide when the income effect is zero:

$$\frac{\partial x_1(p,m)}{\partial m} = 0.$$

In this case, it follows:

$$\frac{\partial x_1(p,m)}{\partial p_1} = \frac{\partial h_1(p,u)}{\partial p_1}$$

A change in price does not affect demand through income, so that the total price effect equals the substitution effect. The two curves (Marshallian and Hicksian) have the same slope and are equal at every price.

For the Cobb-Douglas utility function considered here, the income effect is non-zero, so the Marshallian and Hicksian demands generally differ, unless at the optimum.

3.5 Good type for x_1

Assuming $0 < \alpha < 1$ and taking the derivative wrt income of the Marshallian demand derived in 3.1,

$$\frac{\partial x_1}{\partial m} = \frac{\alpha}{p_1} > 0.$$

Being the income effect positive, it follows that x_1 is a normal good.

4 Welfare (4 points)

4.1 Marshallian and Hicksian demand for x_1 The consumer solves

$$\max_{x_1,x_2} \; 4\sqrt{x_1} + x_2 \quad \text{s.t.} \quad p_1x_1 + p_2x_2 = m.$$

The Lagrangian is

$$\mathcal{L} = 2\sqrt{x_1} + x_2 - \lambda(p_1x_1 + p_2x_2 - m).$$

First order conditions:

$$\frac{\partial \mathcal{L}}{\partial x_1} : \frac{2}{\sqrt{x_1}} - \lambda p_1 = 0,$$
$$\frac{\partial \mathcal{L}}{\partial x_2} : 1 - \lambda p_2 = 0$$

Substitute the x_2 condition, i.e., $\lambda = \frac{1}{p_2}$, into the FOC wrt x_1,

$$\frac{2}{\sqrt{x_1}} = \frac{p_1}{p_2} \quad \Rightarrow \quad \sqrt{x_1} = 2 \cdot \frac{p_2}{p_1} \quad \Rightarrow \quad x_1(\mathbf{p}, m) = \left(\frac{2p_2}{p_1}\right)^2.$$

So the Marshallian demand for x_1 is

$$x_1(\mathbf{p}, m) = \left(\frac{2p_2}{p_1}\right)^2$$

For the Hicksian demand, proceed similarly: Write down the Lagrangian for the EMP, take FOCs, and solve these FOCs for x_1 (denoted by h_1) to reach:

$$h_1(\mathbf{p}, m) = \left(\frac{2p_2}{p_1}\right)^2$$

Alternative solution: Use Slusky's equation and the previous results to say that $x_1(\mathbf{p}, m) = h_1(\mathbf{p}, m)$ given that the income effect is zero.

4.2 Relationship between Marshallian and Hicksian Since the utility function is quasi-linear in x_1, there is no income effect for good x_1. Indeed, using the Marshallian demand derived in 4.1, $\frac{\partial x_1(p,m)}{\partial m} = 0$.

Therefore, the Marshallian and Hicksian demand for x_1 coincide for all prices. As a consequence, consumer surplus, equivalent variation, and compensating variation are equal for any price change.

4.3 Welfare calculations

Plugging x_1 into the budget constraint to find x_2,

$$p_1 \left(\frac{2p_2}{p_1}\right)^2 + p_2 x_2 = m$$
$$x_2(\mathbf{p}, m) = \frac{m}{p_2} - \frac{4p_2}{p_1}.$$

The indirect utility is

$$\begin{aligned} v(\mathbf{p}, , m) &= u\big(x_1(\mathbf{p}, m), x_2(\mathbf{p}, m)\big) \\ &= 4\sqrt{\left(4\frac{p_2}{p_1}\right)^2} + \left(\frac{m}{p_2} - \frac{4p_2}{p_1}\right) \\ &= \frac{8p_2}{p_1} + \frac{m}{p_2} - \frac{4p_2}{p_1} = \frac{m}{p_2} + \frac{4p_2}{p_1}. \end{aligned}$$

To get the expenditure function, let $e(\mathbf{p}, u)$ be the minimum expenditure needed to reach utility u at prices (p_1, p_2). By definition,

$$u = v\big(\mathbf{p}, e(\mathbf{p}, u)\big) = \frac{e(\mathbf{p}, u)}{p_2} + \frac{4p_2}{p_1}.$$

Solve for e:
$$\frac{e(p_1, p_2, u)}{p_2} = u - \frac{4p_2}{p_1} \quad \Rightarrow \quad e(p_1, p_2, u) = p_2 u - \frac{4p_2^2}{p_1}.$$

Take
$$m = 10, \quad p_2 = 4, \quad p_1^0 = 1, \quad p_1^1 = 2.$$

First compute initial utility:
$$v^0 = v(p_1^0, p_2, m) = \frac{m}{p_2} + \frac{4p_2}{p_1^0} = \frac{10}{4} + \frac{16}{1} = \frac{37}{2}.$$

The expenditure needed to reach utility v^0 at the new prices (p_1^1, p_2) is
$$\begin{aligned} e(p_1^1, p_2, v^0) &= p_2 v^0 - 4\frac{p_2^2}{p_1^1} \\ &= 4\frac{37}{2} - \frac{64}{2} = 74 - 32 = 42. \end{aligned}$$

The compensating variation is
$$CV = e(p_1^1, p_2, v^0) - m = 42 - 10 = 32.$$

5 Perfect Competition (2 points)

5.1 In long-run perfect competition with free entry and exit, the equilibrium satisfies:
$$Y(p) = X(p) \qquad \text{and} \qquad \pi_i = 0 \ \forall i.$$

1. Derive the firms' supply function $y_i(p)$ for each firm i.

   Marginal cost is
   $$mc_i(y) = \frac{dc_i(y)}{dy} = y.$$

   Since the supply curve is given by $mc_i(y) = p$, we obtain
   $$y_i(p) = p.$$

2. Derive market supply, which is the sum over all m firms:
   $$Y(p) = \sum_{i=1}^{m} y_i(p) = \sum_{i=1}^{m} p = mp.$$

3. Use the first condition to find the equilibrium price and firm supply as a function of the number of firms.

   Market clearing implies
$$mp = 100 - 2p.$$
   Solving for p,
$$p = \frac{100}{m+2}.$$
   Thus, individual firm supply is
$$y_i(p) = p = \frac{100}{m+2}.$$

4. Use the second condition to find the number of firms m such that profits are zero.

   Profits are
$$\pi_i = py_i(p) - c_i(y).$$
   Substituting and setting equal to zero,
$$\pi_i = \left(\frac{100}{m+2}\right)^2 - \frac{1}{2}\left(\frac{100}{m+2}\right)^2 - 12.5 = 0.$$
   This simplifies to
$$\begin{aligned}\frac{1}{2}\left(\frac{100}{m+2}\right)^2 &= 12.4,\\ \left(\frac{100}{m+2}\right)^2 &= 25,\\ \frac{100}{m+2} &= 5,\\ m &= 18.\end{aligned}$$

Hence, in the long run there will be 18 active firms in this perfectly competitive market.

6 Monopoly (2 points)

6.1 Profit-maximizing level of production (1 point) First, obtain inverse demand from demand function:
$$p(Q) = \frac{45-Q}{3} = 15 - \frac{Q}{3}.$$
Total cost is
$$c(Q) = 45 + 10Q + Q^2 \quad \Rightarrow \quad MC = \frac{\partial TC}{\partial Q} = 10 + 2Q.$$

Marginal revenue:

$$MR = \frac{\partial TR}{\partial Q} = 15 - \frac{2Q}{3}.$$

Profit maximization sets $MR(Q) = MC(Q)$:

$$15 - \frac{2Q}{3} = 10 + 2Q \quad \iff \quad 5 = \frac{8Q}{3} \quad \iff \quad Q^{\star} = \frac{15}{8}$$

6.2 Profit-maximizing level of production (1 point)

A monopolist with a downward sloping demand curve faces a tradeoff. To generate more revenue it needs to sell more, but it can only do so by decreasing the price. When $-1 < \epsilon < 0$, the demand is inelastic, and the monopolist can only sell more if it decreases the price by a lot. When the demand is inelastic, the revenue decreases: the loss from the quantity decrease is too large to compensate for the additional gain. More formally, for a monopolist $MR < 0$ when demand is inelastic:

$$MR(y) = \left(1 + \frac{1}{\epsilon(y)}\right)p(y).$$