Decision Making and Optimization

Linear Programming

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Linear Programming

Model or Formulation

Model or Formulation

Model or Formulation

A model or formulation is a schematic description of a system that takes into account its known properties and can be used to know it better, study its characteristics, and to better program the decisions.

To write a model we must answer the following questions:

- Which are the decisions to take?
- Under which constraints is the decision taken?
- Which is the appropriate criteria that allow an evaluation of the different alternatives of each decision?

Model or Formulation

A typical mathematical programming model or formulation is organized as follow

$$\begin{array}{ll} \text{min or max} & \text{an objective function} \\ \text{subject to:} & \text{a set of constraints} \end{array}$$

with the objective function and constraints being mathematical functions written as a function of the decision variables,

Model or Formulation

defining the decision variables $x \in \mathbb{R}^n$, the mathematical programming model is

$$
\begin{array}{lllr}
\min \text{ (or max)} & z = f(x) & & \\
\text{s. to:} & g_i(x) \leq b_i, & i \in I_1 & (1)\\
 & g_i(x) \geq b_i, & i \in I_2 & (2)\\
 & g_i(x) = b_i, & i \in I_3 & (3)\\
 & & & \\
 & x_j \geq 0, & j \in J_1 & (4)\\
 & x_j \leq 0, & j \in J_2 & (5)\\
 & x_j \in \mathbb{R}, & j \in J_3 & (6)
\end{array}
$$

with

x_j are the decision variables, $x = (x_j)$ is a vector in $\mathbb{R}^n$,
$n = |J| = |J_1 \cup J_2 \cup J_3|$

z is the objective function to be optimised

(1), (2), (3) are the functional constraints, $m = |I| = |I_1 \cup I_2 \cup I_3|$
(4), (5) e (6) are the domain or sign constraints of the variables

Model or Formulation

Some types of mathematical programming models

in case of

- a Linear Programming (LP) model, the objective function and the constraints are all linear
- an Integer Linear Programming (ILP) model, all decision variables are integer
- a Non-Linear Programming (NLP) model, the objective function is nonlinear, the constraints can be linear or nonlinear

Model or Formulation

The General Form of a Linear Programming Model

$$
\begin{array}{lllr}
\text{min or max} & z = \sum_{j\in J} c_j x_j & & \\
\text{s. t.:} & \sum_{j\in J} a_{ij}x_j \leq b_i, & i \in I_1 & (1) \\
 & \sum_{j\in J} a_{ij}x_j \geq b_i, & i \in I_2 & (2) \\
 & \sum_{j\in J} a_{ij}x_j = b_i, & i \in I_3 & (3) \\
 & & & \\
 & x_j \geq 0, & j \in J_1 & (4) \\
 & x_j \leq 0, & j \in J_2 & (5) \\
 & x_j \in \mathbb{R}, & j \in J_3 & (6)
\end{array}
$$

with parameters

c_j the cost coefficient of the decision variable $j \in J$ in the objective function

a_{ij} technological coefficient of the decision variable $j \in J$ in the constraint $i \in I$

b_i the right-hand-side (RHS) of the functional constraint $i \in I$

Model or Formulation

To write a model we must answer the following questions:

- Which are the decisions to take?

  define the decision variables

- Under which constraints is the decision taken?

  define the constraints

- Which is the appropriate criteria that allow an evaluation of the different alternatives of each decision?

  define the objective function

Lets model the challenge

The organic market *From Orchard to Basket* prepares weekly fruit baskets for its customers. There are two types of baskets:

- **Premium Basket (A)** – designed for juices and fruit salads.
- **Smart Basket (B)** – an economical option for daily snacks.

The warehouse has limited fruit stock:

- **Apples:** 10 units available.
- **Oranges:** 14 units available.

Each basket requires the following resources:

- Premium Basket (A): 1 apple and 2 oranges.
- Smart Basket (B): 1 apple and 1 orange.

The profits are:

- Premium Basket (A): 5 € per unit.
- Smart Basket (B): 3 € per unit.

Decide how many baskets A and B to prepare in order to maximize profit without exceeding the stock of fruit.

Assumptions, Definitions and Properties of LP models

Assumptions of LP models

Proportionality: The contribution of each activity j to the value of the objective function and to the left-hand-side of the constraints is proportional to the level x_j of the activity.

Additivity: The value of the objective function and the value of the left-hand-side of the constraints are the sum of the individual contributions of the various activities.

Divisibility: The decision variables $x_j \in \mathbb{R}$ assume real values.

Certainty: Every coefficient (also called parameter) is assumed to be a known constant.

Example

The Reddy Mikks Company (RM) produces both interior and exterior paints from two raw materials, M1 and M2. The necessary quantity of materials (tons of raw material per ton of paint) are displayed in the following table

Product	Interior Paint	Exterior Paint	max daily availability (tons)
Raw Material M1 (tons)	4	6	24
Raw Material M2 (tons)	2	1	6
Profit ($\times$\$1000) / ton	4	5	

The daily demand for interior paint shall not exceed that of exterior paint by more than 1 ton.
The maximum daily demand for interior paint is 2 tons.
RM wants to determine the mix to produce that will maximise the total daily profit.

Example of an LP model with two variables

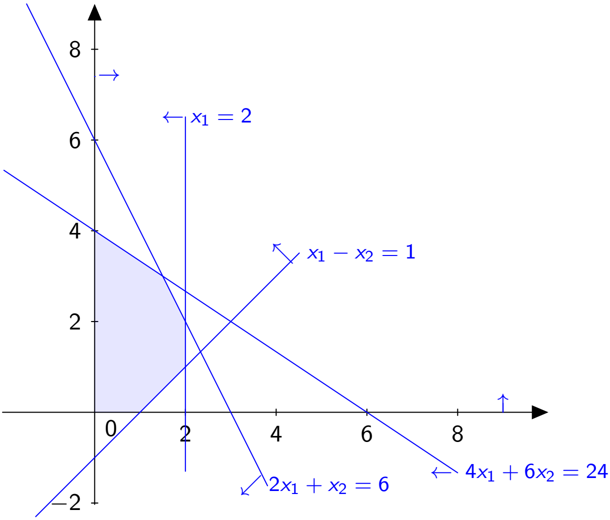
x_1 tons of interior paint produced daily

x_2 tons of exterior paint produced daily

$$\max \quad z = 4x_1 + 5x_2, \quad \times \$1000 \ \leftarrow \text{total daily profit}$$

$$\begin{aligned}
\text{s. to:}\quad & 4x_1 + 6x_2 \leq 24, && \leftarrow \text{raw material M1}\\
& 2x_1 + x_2 \leq 6, && \leftarrow \text{raw material M2}\\
& x_1 - x_2 \leq 1, && \leftarrow \text{interior paint} \leq \text{exterior paint} + 1 \text{ ton}\\
& x_1 \qquad \leq 2, && \leftarrow \text{max daily demand of interior paint is 2 tons}\\
& x_1, x_2 \geq 0, && \leftarrow \text{no negative production}
\end{aligned}$$

The graphical representation of the feasible region



Definitions

Solution of an LP: a vector of $\mathbb{R}^n$ which components are the values of the decision variables

Feasible Solution (FS): a solution that satisfies all the constraints (functional and sign)

Non Feasible Solution (NFS): a solution that does not satisfy at least one of the constraints

Feasible Region (FR): the set of all feasible solutions

Binding constraint in a solution: a constraint that holds with equality at that solution

Optimal Solution (OS): a feasible solution that gives the best value to the objective function (OF) (the best value=maximum or minimum)

Optimal value: the value of the objective function at an optimal solution

Solve an LP

There are several methods to solve an LP

- Graphical method (only for models with two variables)
- Simplex method
- Interior points methods
- ...

There are several software programs that use these methods to solve a LP:

- EXCEL with the Add-in Solver (only for small models)
- solvers such as XPRESS, CPLEX, Gurobi, COIN-OR CLP/CBC, GLPK, MOSEK, CHOCO, MIPCL, HiGHS, SCIP/FSCIP
- modules that help the interaction with solvers such as PuLP a modeler written in Python,
- Languages that help the interaction with solvers Julia, Pyomo
- ...

Solving with the graphical method

Graphical method

1. Graphically represent the set of feasible solutions, identifying all the constraints and the half-space defined by each constraint. The feasible region is the intersection of the half-space defined by all the constraints (functional and sign).
2. If the feasible region is empty, the problem is infeasible. STOP.
3. Graph the gradient of the objective function and the line corresponding to the objective function at any arbitrary point. Set $z = a$ (a fixed and arbitrarily selected) and find the best value a such that the line $z = a$ intersects the feasible region.
    1. If one exists, identify the optimal solution, or set of optimal solutions, from among all the feasible solutions.
    2. Otherwise, conclude the problem is unbounded (there is no optimal solution).

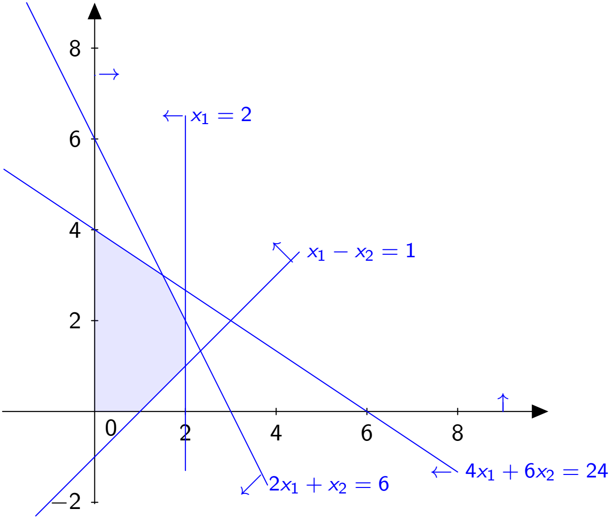
Example of the graphical method: the RM model

Use the graphical method to find the optimal solutions of the following LP, the Reddy Mikks model.

$$
\begin{aligned}
\max \quad & z = 4x_1 + 5x_2, \\
\text{s. to:} \quad & 4x_1 + 6x_2 \leq 24, \\
& 2x_1 + x_2 \leq 6, \\
& x_1 - x_2 \leq 1, \\
& x_1 \qquad \leq 2, \\
& x_1, x_2 \geq 0
\end{aligned}
$$

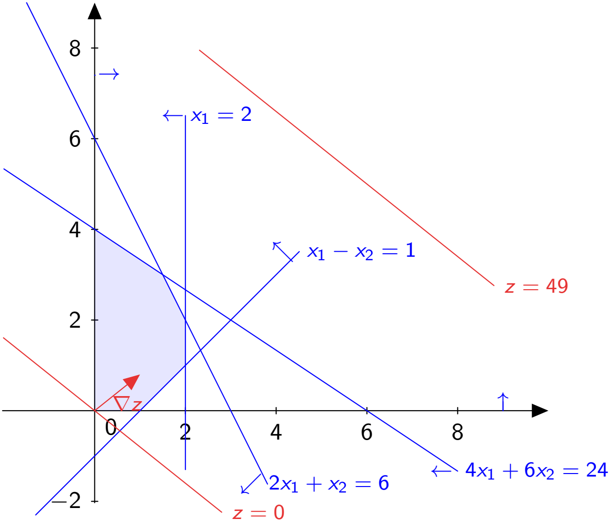
Example of the graphical method: the RM model

Step 1



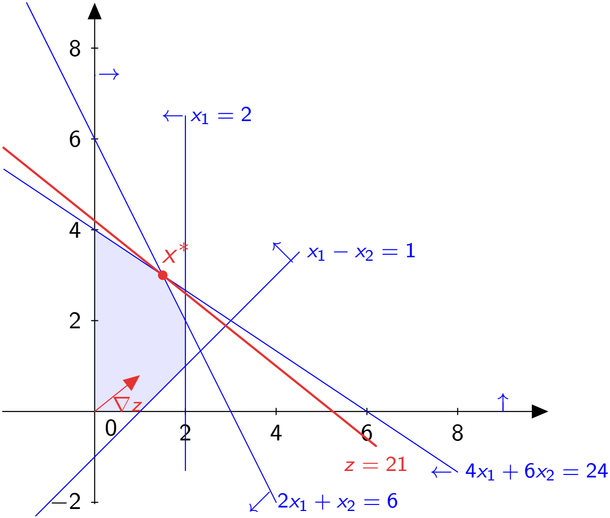
Example of the graphical method: the RM model

Step 2



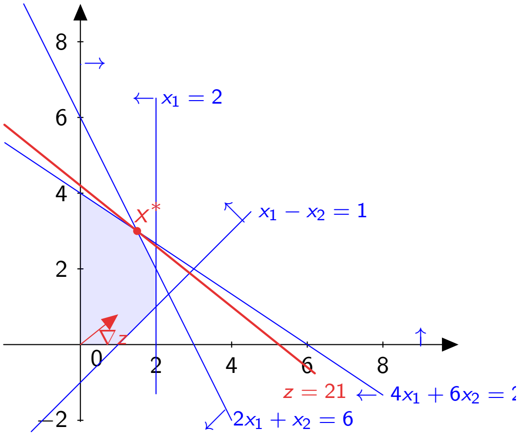
Example of the graphical method: the RM model

Step 3



Example of the graphical method: the RM model

$\nabla z = (4, 5)$
the optimal solution is $x^* = (\frac{3}{2}, 3) = (1.5, 3)$ with value $z^* = 21$



Properties

Property 1

The feasible region of an LP problem is either an empty set or a convex set.

Property 2

If the feasible region of an LP problem is nonempty and bounded, then there exists an optimal solution.

Property 3

If an LP problem has an optimal solution, then at least one of the extreme points of the feasible region is an optimal solution.

Property 4

Given an LP problem with an optimal solution, if an extreme point of the feasible region has no adjacent extreme points with a better value for the objective function, then that extreme point is an optimal solution.

Exercise: model and graphically solve

Ozark Farms uses at least 800 Kg of special feed daily. The special feed is a mixture of corn and soybean meal with the following compositions:

feedstuff	protein (Kg)	fiber (Kg)	Cost ($/Kg)
Corn	0.09	0.02	0.30
Soybean meal	0.60	0.06	0.90

Table: Components (Kg) and Cost ($) per Kg of feedstuff

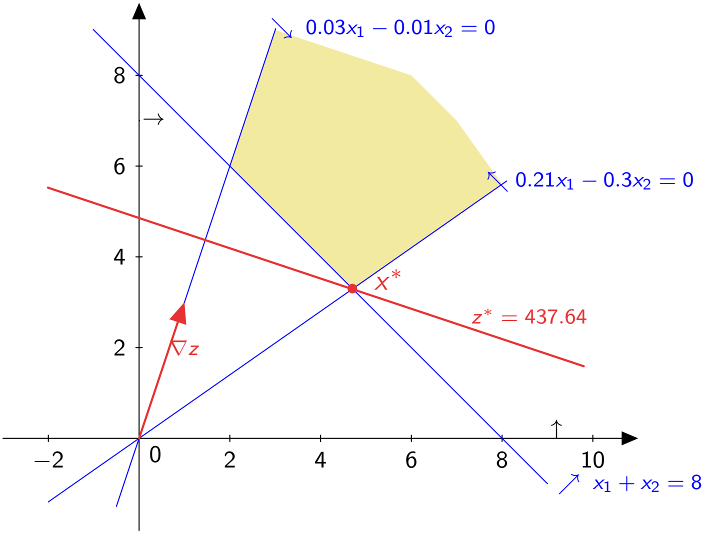
The dietary requirements of special feed are at least 30% protein and at most 5% fiber. The goal is to determine the daily minimum-cost feed mix.

Model

x_1 Kg of corn in the daily mix

x_2 Kg of soybean meal in the daily mix

$$
\begin{aligned}
\min \quad & z = 0.3x_1 + 0.9x_2, && \leftarrow \text{total daily cost (\$)} \\
\text{s. to:} \quad & x_1 + x_2 \geq 800, && \leftarrow \text{Kg of feed required daily} \\
& 0.21x_1 - 0.3x_2 \leq 0, && \leftarrow \text{protein requirement, min 30\%} \\
& 0.03x_1 - 0.01x_2 \geq 0, && \leftarrow \text{fibber requirement, max 5\%} \\
& x_1, x_2 \geq 0, && \leftarrow \text{no negative amounts}
\end{aligned}
$$



$x^* = (470.6; 329.4)$ and $z^* = 437.64$

Solve an LP: find optimal solutions, when exist

To solve an LP is to determine the optimal solution (or solutions) and the optimal value or to conclude that an optimal solution does not exist and why.

- If the feasible region is an empty set, then the problem is impossible (see below Example 3).
- If the feasible region is a non-empty set, then the problem is possible,
  - if there is an optimal solution, find its value and all its (alternative) optimal solutions,
    - the problem can have a unique optimal solution (see below Examples 1, 2, 5),
    - the problem can have alternative optimal solutions (in this case it has an infinite number of optimal solutions) (see below Examples 6, 7),
  - otherwise, conclude that the problem has an objective function with unlimited value, the problem is unbounded (see below Example 4).

Solve an LP: optimal solutions

a non-empty feasible region of an LP is always a convex set, it can be bounded or unbounded

types of optimal solutions of an LP problem

- unique optimal solution
  - is an extreme point of the feasible set (see below examples 1, 2, 5)
- alternative optimal solutions
  - linear convex combination of extreme points of the feasible set (see below example 6)
  - if the feasible set is unbounded, it can also be a linear convex combination of extreme points of the feasible set plus a linear positive combination of extreme directions of the feasible set (see below example 7)

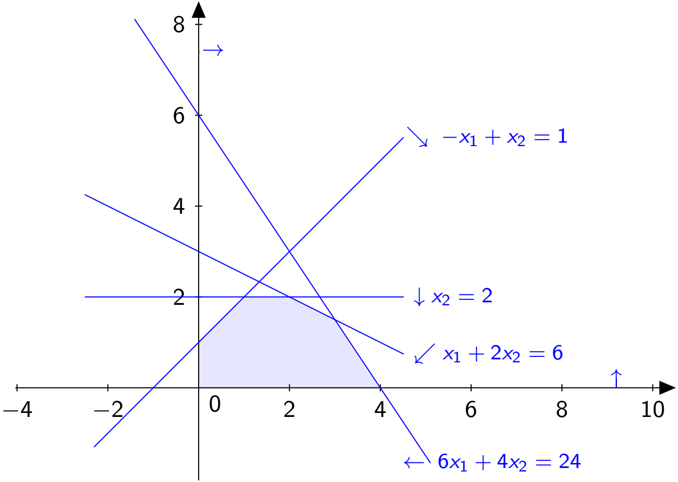
Example 1

Use the graphical method to identify the optimal solutions of the following LP.

$$
\begin{aligned}
\max \quad & z = 5x_1 + 4x_2, \\
\text{s. to:} \quad & 6x_1 + 4x_2 \leq 24, \\
& x_1 + 2x_2 \leq 6, \\
& -x_1 + x_2 \leq 1, \\
& x_2 \leq 2, \\
& x_1, x_2 \geq 0
\end{aligned}
$$

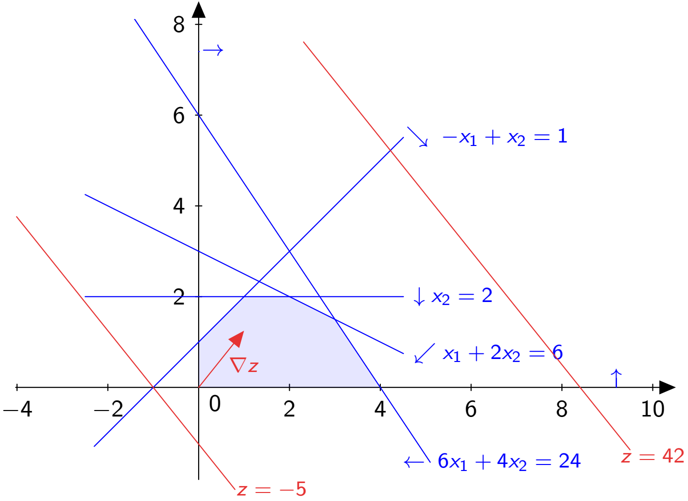
Example 1

Step 1



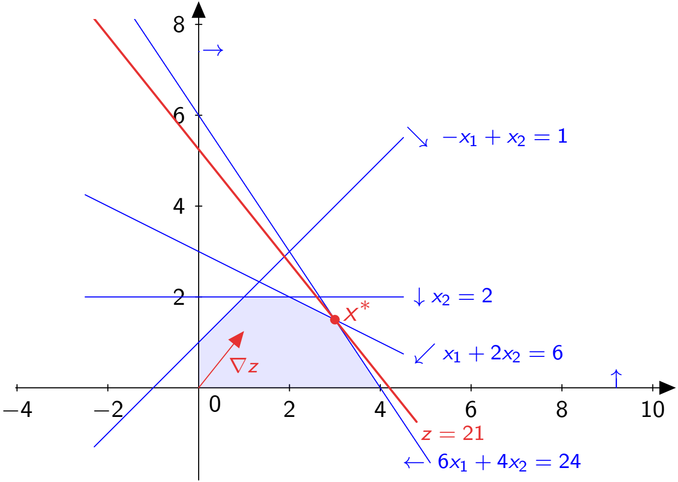
Example 1

Step 2



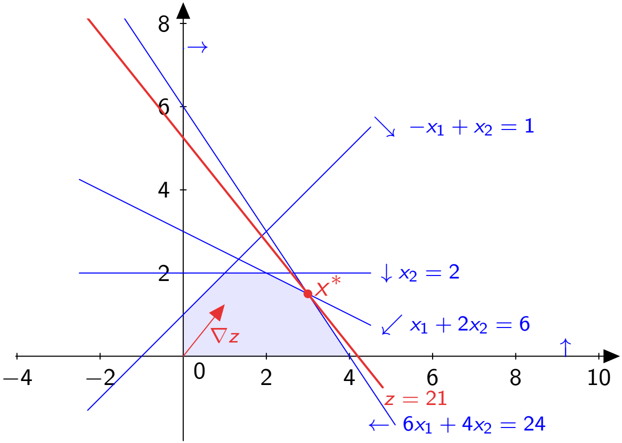
Example 1

Step 3



Example 1

unique optimal solution, the optimal solution is
$x^* = (3, \frac{3}{2}) = (3; 1.5)$ with value $z^* = 21$



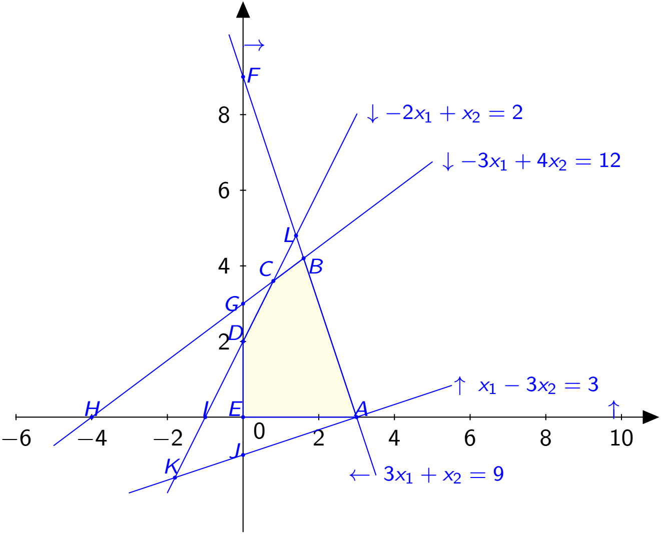
Example 2

Use the graphical method to identify the optimal solutions of the following LP.

$$
\begin{aligned}
\max\ & Z = x_1 + 3x_2 \\
\text{s. to}\quad & x_1 - 3x_2 \leq 3 \\
& -2x_1 + x_2 \leq 2 \\
& -3x_1 + 4x_2 \leq 12 \\
& 3x_1 + x_2 \leq 9 \\
& x_1,\ x_2 \geq 0
\end{aligned}
$$

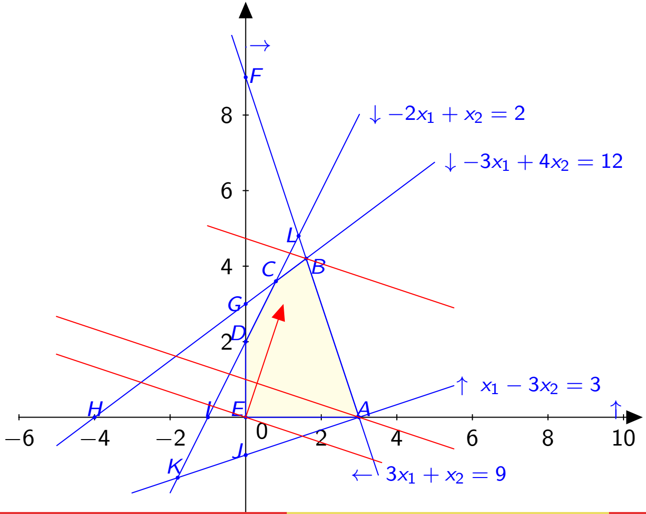
Sketch the feasible region in the space of variables $\{x_1, x_2\}$ and identify the optimal solution.

Example 2


F
$\downarrow -2x_1 + x_2 = 2$
$\downarrow -3x_1 + 4x_2 = 12$
L
B
C
G
D
E
A
H
I
J
K
$\uparrow\ x_1 - 3x_2 = 3$
$\leftarrow 3x_1 + x_2 = 9$
8
6
4
2
0
−6
−4
−2
2
4
6
8
10

Example 2

the gradient of the objective function is the vector $\nabla Z = (1, 3)$
the unique optimal solution is B

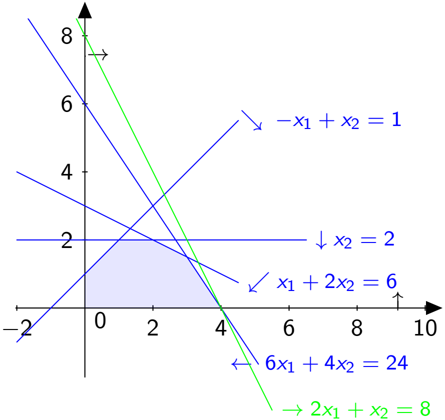


Example 3

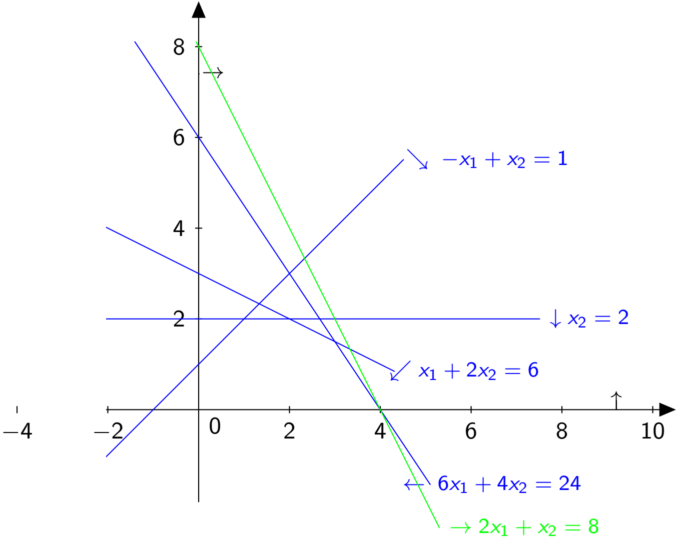
Use the graphical method to find the optimal solutions of the following LP.

$$\max \quad z = 5x_1 + 4x_2,$$

$$\begin{aligned} \text{s. to:} \quad & 6x_1 + 4x_2 \leq 24, \\ & x_1 + 2x_2 \leq 6, \\ & -x_1 + x_2 \leq 1, \\ & x_2 \leq 2, \\ & \color{green}{2x_1 + x_2 \geq 8,} \\ & x_1, x_2 \geq 0 \end{aligned}$$



Example 3



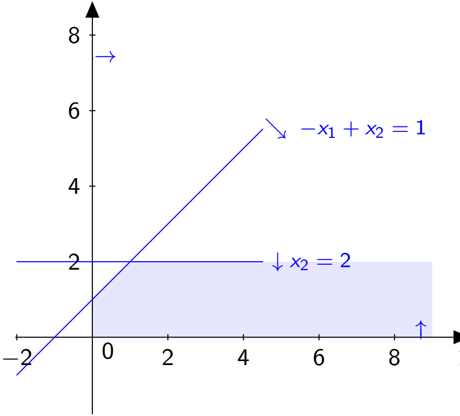
we obtain an empty feasible set, thus the problem is impossible

Example 4

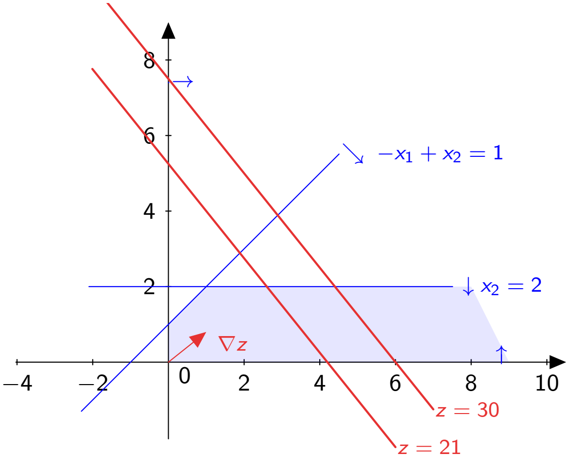
Use the graphical method to identify the optimal solutions of the following LP.

$$\max \quad z = 5x_1 + 4x_2,$$

$$\begin{aligned} \text{s. to:} \quad -x_1 + x_2 &\leq 1, \\ x_2 &\leq 2, \\ x_1, x_2 &\geq 0 \end{aligned}$$



Example 4



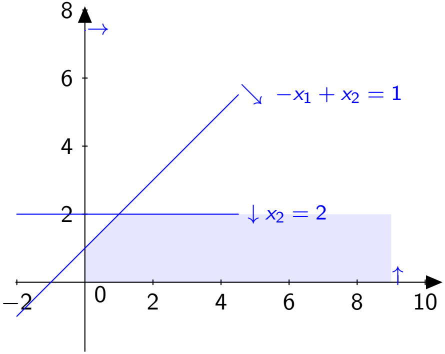
the feasible region is unbounded and the problem is unlimited

Example 5

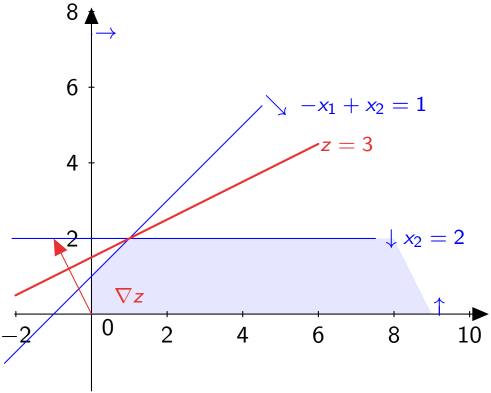
Use the graphical method to identify the optimal solutions of the following LP.

$$\max \quad z = -x_1 + 2x_2,$$

$$\text{s. to:} \quad -x_1 + x_2 \leq 1,$$
$$x_2 \leq 2,$$
$$x_1, x_2 \geq 0$$



Example 5

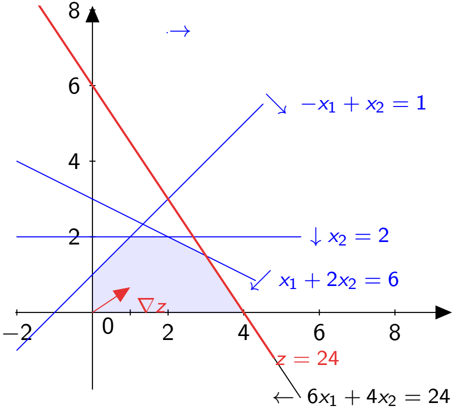


feasible region is unbounded and the unique optimal solution is $(1, 2)$

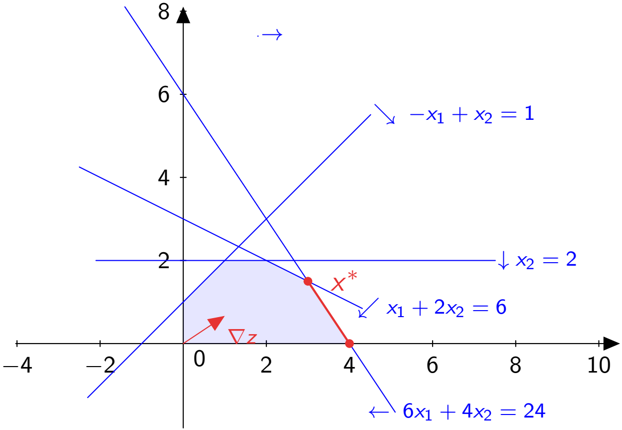
Example 6

Use the graphical method to identify the optimal solutions of the following LP.

$$
\begin{aligned}
\max \quad & z = 6x_1 + 4x_2, \\
\text{s. to:} \quad & 6x_1 + 4x_2 \leq 24, \\
& x_1 + 2x_2 \leq 6, \\
& -x_1 + x_2 \leq 1, \\
& x_2 \leq 2, \\
& x_1, x_2 \geq 0
\end{aligned}
$$



Example 6



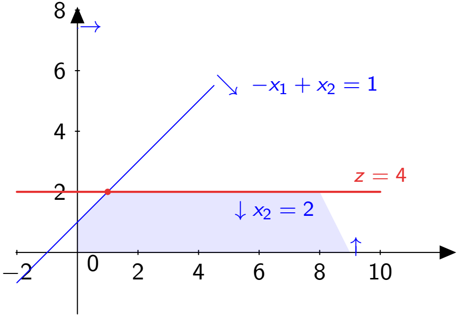
alternative optimal solutions $x^* = \alpha(3, \frac{3}{2}) + (1-\alpha)(4, 0),\ \alpha \in [0, 1]$, the line segment between $(3, \frac{3}{2})$ and $(4, 0)$

Example 7

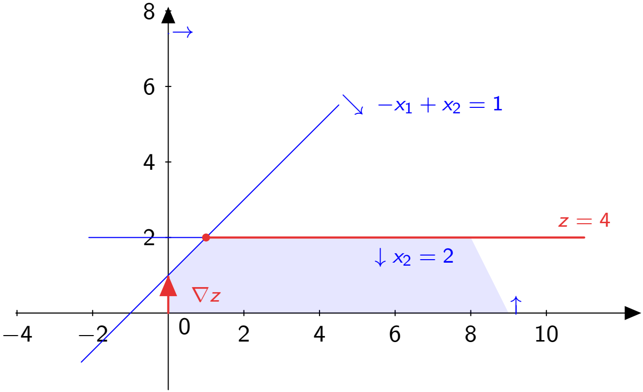
Use the graphical method to identify the optimal solutions of the following LP.

$$\max \quad z = 2x_2,$$

$$\begin{aligned} \text{s. to:} \quad -x_1 + x_2 &\leq 1, \\ x_2 &\leq 2, \\ x_1, x_2 &\geq 0 \end{aligned}$$



Example 7



feasible region is unbounded and the alternative optimal solutions are $x^* = (1, 2) + \beta(1, 0),\ \beta \geq 0$, the semi-line starting at point $(1, 2)$ and direction of vector $(1, 0)$

Solving with the Excel Solver

Excel Solver

The software Excel, with the Add-in Solver, can easily be used to solve small LP problems.

The first time, make sure you have the Solver add-in installed in the Excel software.

Go to: File (at the top left) $\rightarrow$ Options (at the bottom) $\rightarrow$ Add-ins (on the left panel) $\rightarrow$ Manage: Excel Add-ins (at the bottom) $\rightarrow$ Go $\rightarrow$ Select the Solver Add-in $\rightarrow$ OK.

Excel Solver

1. Identify a column for each decision variable.
2. Identify a row for the objective function and for each constraint.
3. Insert all the data:
   1. coefficients in the objective function, in the corresponding row and column associated to each variable;
   2. technological coefficients in each corresponding constraint row and column associated to each variable;
   3. for each constraint, also insert the corresponding RHS in a separate column.
4. Select the cells where you want the Excel Solver to display the optimal values of the decision variables.

Excel Solver

5. Select a cell where you want the Excel Solver to display the value of the optimal solution. Write in the cell, using the SUMPRODUCT function of the Excel, the objective function, using the cells of the coefficients and the cells selected to display the optimal values of the decision variables.
6. For each constraint, write in a separate cell, using the SUMPRODUCT function of the Excel, the constraint function, using the cells of the coefficients and the cells selected to display the optimal values of the decision variables.

Excel Solver

7. Go to the Data panel at the top, select the solver (that you have installed) from the available operations.
    1. Make sure the cell you selected to display the value of the optimal solution, corresponds to the one in the box Set Objective.
    2. Select the Max or Min criteria for your problem.
    3. Make sure the cells you selected to display the optimal value of the decision variables appear in the By Changing Variable Cells box.
    4. Use the Add botton to add all the constraints by selecting: the cell where the constraint function is written, the cell where the rhs is located, and select the sign of the inequality.
    5. Select a solving method: Simplex LP
    6. Press the Solve botton.
    7. Select the Reports and press OK.

Example of the Excel Solver

Use the Excel Solver to find the optimal solutions of the following LP.

$$\begin{aligned} \max \quad & z = 4x_1 + 5x_2, \\ \text{s. to:} \quad & 4x_1 + 6x_2 \leq 24, \\ & 2x_1 + x_2 \leq 6, \\ & x_1 - x_2 \leq 1, \\ & x_1 \qquad \leq 2, \\ & x_1, x_2 \geq 0 \end{aligned}$$

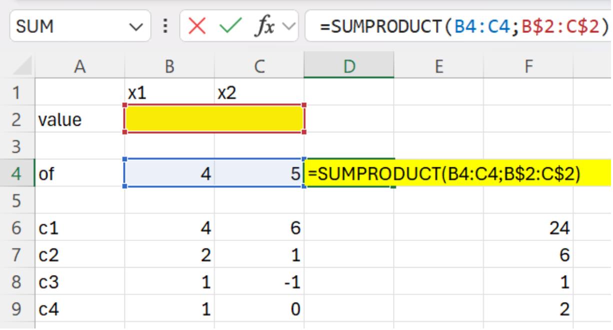
Example of the Excel Solver

Steps 1, 2, and 3

	A	B	C	D	E	F
1		x1	x2			
2	value					
3						
4	of	4	5			
5						
6	c1	4	6			24
7	c2	2	1			6
8	c3	1	-1			1
9	c4	1	0			2

Example of the Excel Solver

Steps 4, and 5

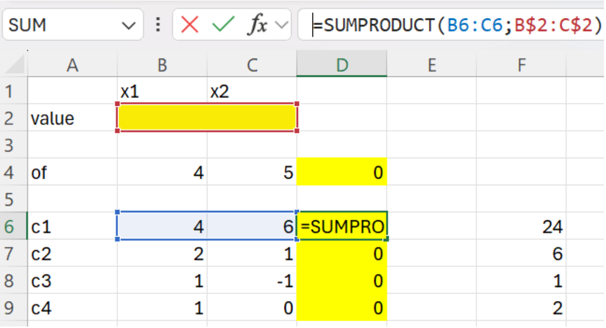


SUM | =SUMPRODUCT(B4:C4;B$2:C$2)

	A	B	C	D	E	F
1		x1	x2			
2	value					
3						
4	of	4	5	=SUMPRODUCT(B4:C4;B$2:C$2)		
5						
6	c1	4	6			24
7	c2	2	1			6
8	c3	1	-1			1
9	c4	1	0			2

Example of the Excel Solver

Step 6, copy the formula for all the constraints

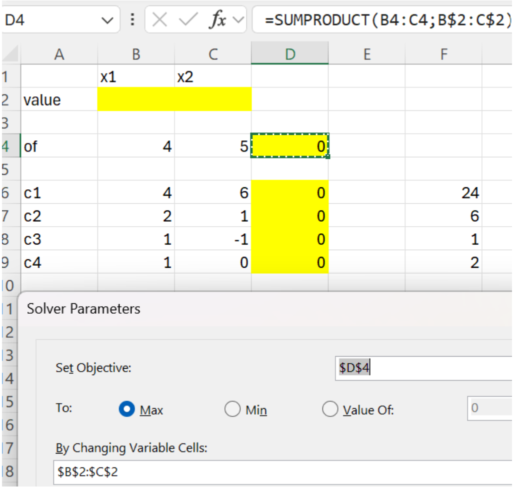


SUM | =SUMPRODUCT(B6:C6;B$2:C$2)

	A	B	C	D	E	F
1		x1	x2			
2	value					
3						
4	of	4	5	0		
5						
6	c1	4	6	=SUMPRO		24
7	c2	2	1	0		6
8	c3	1	-1	0		1
9	c4	1	0	0		2

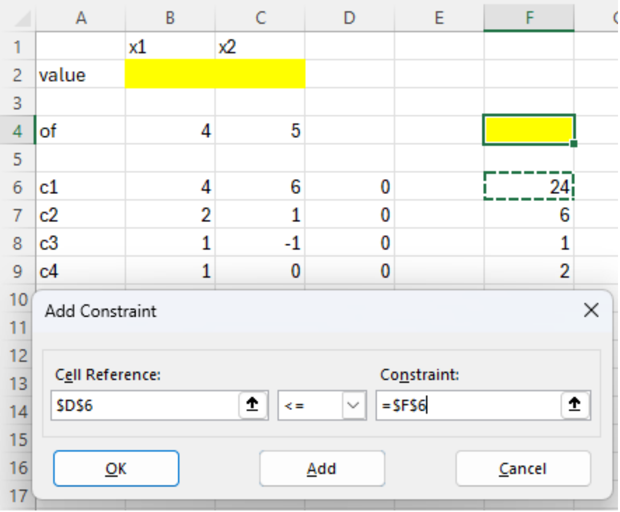
Example of the Excel Solver

Step 7 - 1,2,3



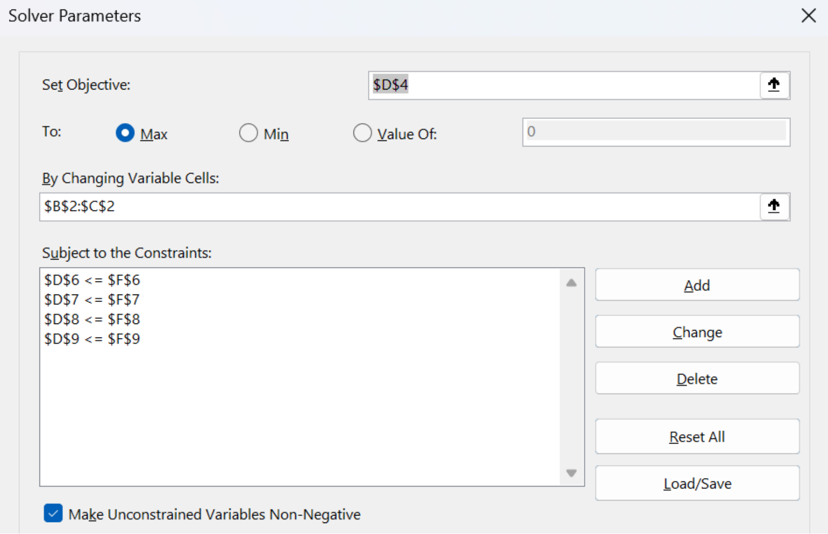
Example of the Excel Solver

Step 7 - 4



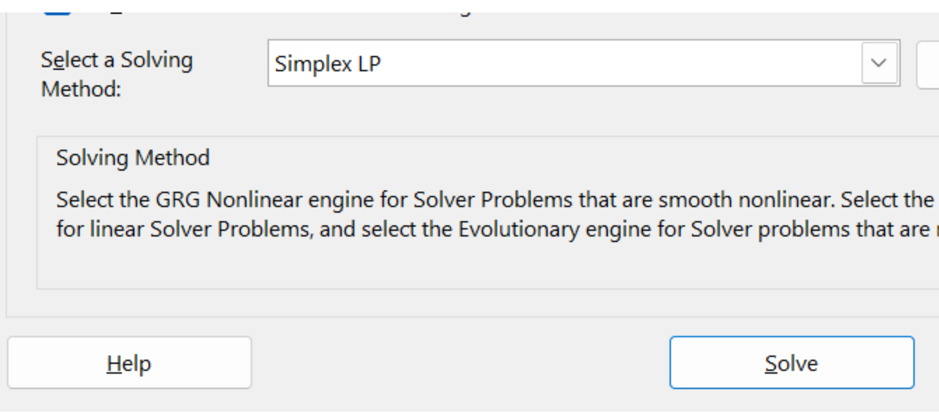
Example of the Excel Solver

Step 7 - 1,2,3,4



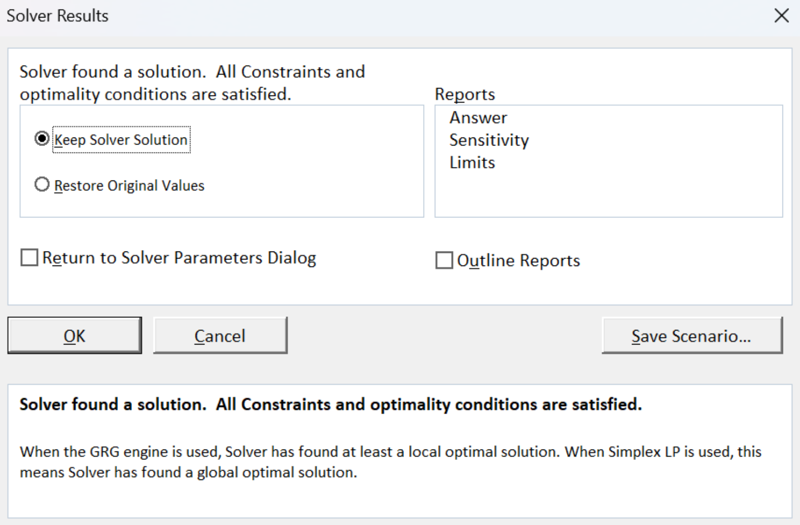
Example of the Excel Solver

Step 7 - 5, 6



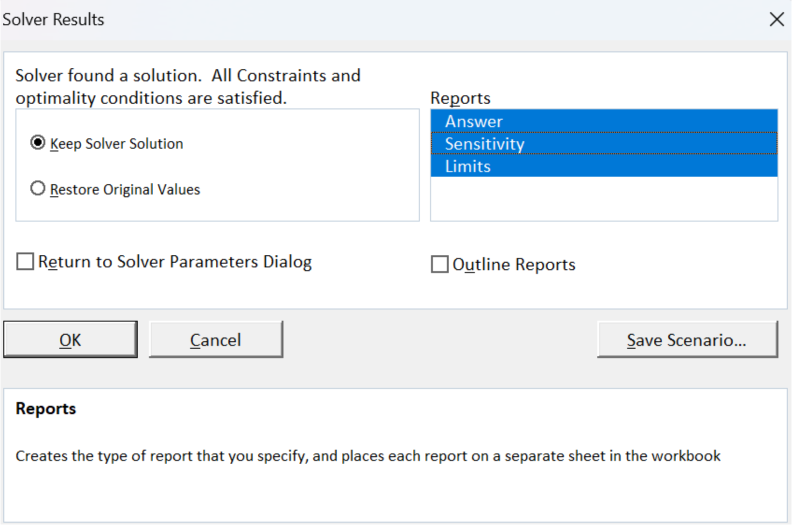
Example of the Excel Solver

Step 7 - after 6



Example of the Excel Solver

Step 7 - 7



Example of the Excel Solver

The results obtained

	A	B	C	D	E	F
1		x1	x2			
2	value	1,5	3			
3						
4	of	4	5	21		
5						
6	c1	4	6	24		24
7	c2	2	1	6		6
8	c3	1	-1	-1,5		1
9	c4	1	0	1,5		2
10						

Example of the Excel Solver

The Answer Report

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$D$4	of	0	21

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$2	value x1	0	1,5	Contin
$C$2	value x2	0	3	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$D$6	c1	24	$D$6<=$F$6	Binding	0
$D$7	c2	6	$D$7<=$F$7	Binding	0
$D$8	c3	-1,5	$D$8<=$F$8	Not Binding	2,5
$D$9	c4	1,5	$D$9<=$F$9	Not Binding	0,5

Example of the Excel Solver

The Sensitivity Report

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
\$B\$2	value x1	1,5	0	4	6	0,666666667
\$C\$2	value x2	3	0	5	1	3

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
\$D\$6	c1	24	0,75	24	12	4
\$D\$7	c2	6	0,5	6	0,666666667	2
\$D\$8	c3	-1,5	0	1	1E+30	2,5
\$D\$9	c4	1,5	0	2	1E+30	0,5

Exercise: solve with Excel Solver

Ozark Farms uses at least 800 Kg of special feed daily. The special feed is a mixture of corn and soybean meal with the following compositions:

feedstuff	protein (Kg)	fiber (Kg)	Cost ($/Kg)
Corn	0.09	0.02	0.30
Soybean meal	0.60	0.06	0.90

Table: Components (Kg) and Cost ($) per Kg of feedstuff

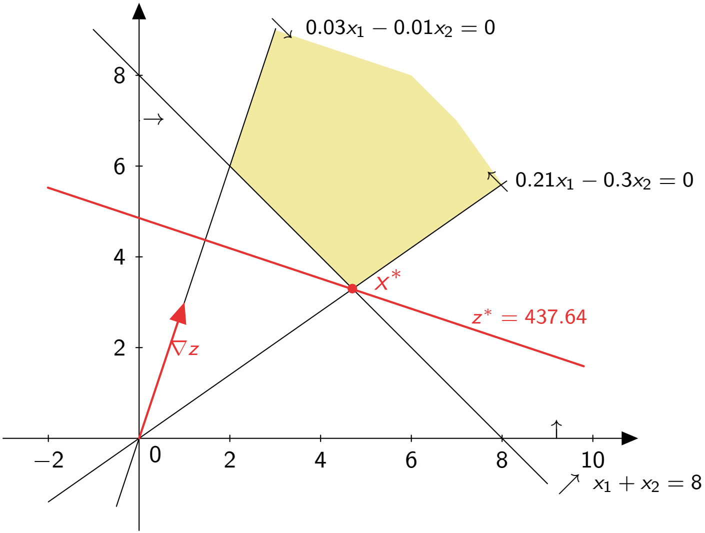
The dietary requirements of special feed are at least 30% protein and at most 5% fiber. The goal is to determine the daily minimum-cost feed mix.

Model

x_1 Kg of corn in the daily mix

x_2 Kg of soybean meal in the daily mix

$$
\begin{array}{lll}
\min & z = 0.3x_1 + 0.9x_2, & \leftarrow \text{total daily cost (\$)} \\
\text{s. to:} & x_1 + x_2 \geq 800, & \leftarrow \text{Kg of feed required daily} \\
& 0.21x_1 - 0.3x_2 \leq 0, & \leftarrow \text{protein requirement, min 30\%} \\
& 0.03x_1 - 0.01x_2 \geq 0, & \leftarrow \text{fibber requirement, max 5\%} \\
& x_1, x_2 \geq 0, & \leftarrow \text{no negative amounts}
\end{array}
$$



$x^* = (470.6; 329.4)$ and $z^* = 437.64$

Answer Report from Excel Solver

Objective Cell (Min)

Cell	Name	Original Value	Final Value
$E$4	z	0	437,647

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$C$10	x1	0	470,588	Contin
$D$10	x2	0	329,412	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$E$6	C1	800	$E$6>=$G$6	Binding	0
$E$7	C2	-1,429E-14	$E$7>=$G$7	Binding	0
$E$8	C3	10,824	$E$8>=$G$8	Not Binding	10,824

Solving with the software XPRESS

Example of the XPRESS

Use the software FICO XPRESS to find the optimal solutions of the following LP. Learn more here.

$$
\begin{aligned}
\max \quad & z = 4x_1 + 5x_2, \\
\text{s. to:} \quad & 4x_1 + 6x_2 \leq 24, \\
& 2x_1 + x_2 \leq 6, \\
& x_1 - x_2 \leq 1, \\
& x_1 \qquad \leq 2, \\
& x_1, x_2 \geq 0
\end{aligned}
$$

Example of the XPRESS

```
model RM
uses "mmxprs", "mmsystem"
declarations
   m: integer
   n: integer
end-declarations
!define the data - matrix dimension
m = 4
n = 2
declarations
   M=1..m
   N=1..n
   x: array(N)of mpvar
   z: linctr
   A: array(M,N) of integer
   B: array(M) of integer
   C: array(N) of integer
end-declarations
!define A, B, C
A :: [6, 4, 1, 2, -1, 1, 0, 1 ]
B :: [24, 6, 1, 2]
C :: [5, 4]
!define the model
!objective function:
z:= sum(i in N) C(i)*x(i)

!constraints:
forall(j in M) Constraint(j):= sum(i in N)
A(j,i)*x(i) <= B(j)

!define the variables domain:
forall (i in N) x(i) >=0

maximize(z) !solve problem

writeln("Optimal Solution:")

writeln(" Objective function value z =
",getobjval)

forall(i in N)
   writeln(" x(",i,")= ", getsol(x(i)))

end-model
```

Solving using Python with the PuLP solver

Example using Python with PuLP

Use the Python language with the modeler PuLP to find the optimal solutions of the following LP.

$$\begin{aligned} \max \quad & z = 4x_1 + 5x_2, \\ \text{s. to:} \quad & 4x_1 + 6x_2 \leq 24, \\ & 2x_1 + x_2 \leq 6, \\ & x_1 - x_2 \leq 1, \\ & x_1 \qquad \leq 2, \\ & x_1, x_2 \geq 0 \end{aligned}$$

Example using Python with PuLP

```
# import the solver pulp
from pulp import *
# define the data
m = 4
n = 2
indexI = range(1,m+1)
indexJ = range(1,n+1)
# define A, B, C
A = [ [6, 4], [1, 2], [-1, 1], [0, 1] ]
B = [24, 6, 1, 2]
C = [5, 4]
A = makeDict([indexI, indexJ], A)
B = makeDict([indexI], B)
C = makeDict([indexJ], C)

# define the model
model_RM = LpProblem("RM", LpMaximize)
```

```
# define the variables
x = LpVariable.dicts("x", indexJ, 0, None)

# objective function
model_RM += (lpSum([x[j] * C[j] for j in
indexJ]), "FO", )

# constraints
for i in indexI: model_RM += (
lpSum([A[i][j]*x[j] for j in indexJ]) <= B[i],
f"constraint_i", )

model_RM.solve() # solve the problem

status=LpStatus[model_RM.status]
print("Status:", LpStatus[model_RM.status])

# write the solution
print("OF value = ",
value(model_RM.objective))
for v in model_RM.variables():
print(v.name, "=", v.varValue)
```

Standard, Augmented and Matricial LP Forms

Standard Minimization LP Form

$$
\begin{array}{lll}
\min & z = \sum_{j=1}^{n} c_j x_j & \\
\text{s. to:} & & \\
& \sum_{j=1}^{n} a_{ij} x_j \geq b_i, & i = 1, \ldots, m \\
& x_j \geq 0, & j = 1, \ldots, n
\end{array}
$$

Standard Maximization LP Form

$$
\begin{aligned}
\max \quad & z = \sum_{j=1}^{n} c_j x_j \\
\text{s. to:} \quad & \\
& \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, \ldots, m \\
& x_j \geq 0, \quad j = 1, \ldots, n
\end{aligned}
$$

Reformulation operations on constraints

an inequality can be transformed into an opposite, in sign, inequality

$$\sum_{j=1}^{n} a_{ij}x_j \geq b_i \qquad \Leftrightarrow \qquad \sum_{j=1}^{n} -a_{ij}x_j \leq -b_i$$

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i \qquad \Leftrightarrow \qquad \sum_{j=1}^{n} -a_{ij}x_j \geq -b_i$$

Reformulation operations on constraints

an equality is always equivalent to two inequalities

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \qquad \Leftrightarrow \qquad \begin{cases} \sum_{j=1}^{n} a_{ij}x_j \leq b_i \\ \sum_{j=1}^{n} a_{ij}x_j \geq b_i \end{cases}$$

Reformulation operations on constraints

an inequality can be transformed into an equality by including a non-negative slack variable, $x_i^s \geq 0$,

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i \quad \Leftrightarrow \quad \begin{cases} \sum_{j=1}^{n} a_{ij} x_j - x_i^s = b_i \\ x_i^s \geq 0 \end{cases}$$

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad \Leftrightarrow \quad \begin{cases} \sum_{j=1}^{n} a_{ij} x_j + x_i^s = b_i \\ x_i^s \geq 0 \end{cases}$$

Reformulation operations on the domain constraints

usually the domain constraints of the variables are non-negativity constraints, replace variable x_j by x_j' (and x_j'')

$$x_j \leq 0 \quad \rightarrow \quad \begin{cases} x_j = -x_j' \\ x_j' \geq 0 \end{cases}$$

$$x_j \in \mathbb{R} \quad \rightarrow \quad \begin{cases} x_j = x_j' - x_j'' \\ x_j', x_j'' \geq 0 \end{cases}$$

$$x_j \geq l_j \quad \rightarrow \quad \begin{cases} x_j' = x_j - l_j \\ x_j' \geq 0 \end{cases}$$

$$x_j \leq u_j \quad \rightarrow \quad \begin{cases} x_j' = u_j - x_j \\ x_j' \geq 0 \end{cases}$$

Reformulation operations on the objective function

$$max\ z = -min - z$$

$$min\ z = -max - z$$

Augmented LP Form

$$
\begin{aligned}
\min/\max \quad & z = \sum_{j=1}^{n} c_j x_j \\
\text{s. t.:} \quad & \\
& \sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i = 1, \ldots, m \\
& x_j \geq 0, \qquad\qquad j = 1, \ldots, n
\end{aligned}
$$

m constraints

n variables: decision variables plus slack variables

Matricial Augmented Form

$$\begin{array}{ll} \max & z = c^T x \\ \text{s. t.:} & Ax = b \\ & x \geq 0 \end{array}$$

with

- $x \in \mathbb{R}^n$, n variables in total (decision variables plus slack variables)
- m constraints (technological)
- $x \in \mathbb{R}^n$
- $A_{m \times n}$ matrix such that $n > m$ and $rank(A, b) = rank(A) = m$

Example of the augmented form: the RM model

Consider the non-negative slack variables $x_3, x_4, x_5, x_6 \geq 0$, the augmented form of the RM model is

$$
\begin{aligned}
\max \quad & z = 4x_1 + 5x_2, \\
\text{s. t.:} \quad & 4x_1 + 6x_2 + x_3 = 24, \\
& 2x_1 + x_2 + x_4 = 6, \\
& x_1 - x_2 + x_5 = 1, \\
& x_1 + x_6 = 2, \\
& x_1, x_2, x_3, x_4, x_5, x_6 \geq 0
\end{aligned}
$$

Extreme points of an LP

Extreme points

A non-empty feasible region has a finite number of extreme points.

There are:

$m = 4$ technological constraints ($\rightarrow$ 4 slack variables)
$n = 6$ variables (2 decision plus 4 slack)
$n - m = 2$ decision variables

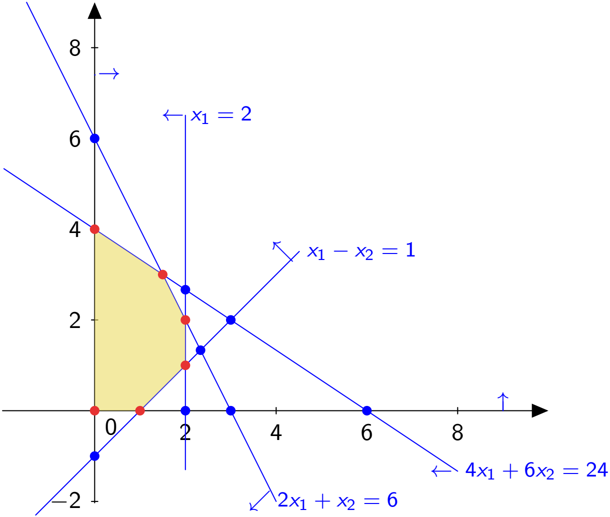
There are at most C^n_{n-m} extreme points

The RM model has at most $C^n_{n-m} = C^6_2 = \frac{6!}{(6-2)!\ 2!} = \frac{6\times 5}{2} = 15$ extreme points

An extreme point is a feasible solution at which at least $n - m$ constraints are binding.

Extreme points of the RM model

Extreme points are: $(0,0)$; $(1,0)$; $(2,1)$; $(2,2)$; $(1.5,3)$; $(0,4)$



How to find extreme points?

Find solutions at which there are $n-m$ binding constraints. For the RM model, $n-m=2$:

$\begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases} \Leftrightarrow (0,0)$	$\begin{cases} x_1 = 0 \\ 4x_1 + 6x_2 = 24 \end{cases} \Leftrightarrow (0,4)$	$\begin{cases} x_1 = 0 \\ 2x_1 + x_2 = 6 \end{cases} \Leftrightarrow (0,6)$
$\begin{cases} x_1 = 0 \\ x_1 - x_2 = 1 \end{cases} \Leftrightarrow (0,-1)$	$\begin{cases} x_2 = 0 \\ 4x_1 + 6x_2 = 24 \end{cases} \Leftrightarrow (6,0)$	$\begin{cases} x_2 = 0 \\ 2x_1 + x_2 = 6 \end{cases} \Leftrightarrow (3,0)$
$\begin{cases} x_2 = 0 \\ x_1 - x_2 = 1 \end{cases} \Leftrightarrow (1,0)$	$\begin{cases} x_1 = 2 \\ x_2 = 0 \end{cases} \Leftrightarrow (2,0)$	$\begin{cases} 4x_1 + 6x_2 = 24 \\ 2x_1 + x_2 = 6 \end{cases} \Leftrightarrow (\frac{3}{2},3)$
$\begin{cases} 4x_1 + 6x_2 = 24 \\ x_1 - x_2 = 1 \end{cases} \Leftrightarrow (3,2)$	$\begin{cases} 4x_1 + 6x_2 = 24 \\ x_1 = 2 \end{cases} \Leftrightarrow (2,\frac{8}{3})$	$\begin{cases} 2x_1 + x_2 = 6 \\ x_1 - x_2 = 1 \end{cases} \Leftrightarrow (\frac{7}{3},\frac{4}{3})$
$\begin{cases} 2x_1 + x_2 = 6 \\ x_1 = 2 \end{cases} \Leftrightarrow (2,2)$	$\begin{cases} x_1 - x_2 = 1 \\ x_1 = 2 \end{cases} \Leftrightarrow (2,1)$	$C_2^6 = \frac{6!}{(6-2)!\ 2!} = \frac{6\times 5}{2} = 15$

The extreme points are the ones that are feasible.

How to find extreme points?

Or, equivalently, find solutions having, at least, $n - m$ variables equal zero. For the RM model, $n - m = 2$:

$\begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases} \Leftrightarrow (0, 0)$	$\begin{cases} x_1 = 0 \\ x_3 = 0 \end{cases} \Leftrightarrow (0, 4)$	$\begin{cases} x_1 = 0 \\ x_4 = 0 \end{cases} \Leftrightarrow (0, 6)$
$\begin{cases} x_1 = 0 \\ x_5 = 0 \end{cases} \Leftrightarrow (0, -1)$	$\begin{cases} x_2 = 0 \\ x_3 = 0 \end{cases} \Leftrightarrow (6, 0)$	$\begin{cases} x_2 = 0 \\ x_4 = 0 \end{cases} \Leftrightarrow (3, 0)$
$\begin{cases} x_2 = 0 \\ x_5 = 0 \end{cases} \Leftrightarrow (1, 0)$	$\begin{cases} x_2 = 0 \\ x_6 = 0 \end{cases} \Leftrightarrow (2, 0)$	$\begin{cases} x_3 = 0 \\ x_4 = 0 \end{cases} \Leftrightarrow (\frac{3}{2}, 3)$
$\begin{cases} x_3 = 0 \\ x_5 = 0 \end{cases} \Leftrightarrow (3, 2)$	$\begin{cases} x_3 = 0 \\ x_6 = 0 \end{cases} \Leftrightarrow (2, \frac{8}{3})$	$\begin{cases} x_4 = 0 \\ x_5 = 0 \end{cases} \Leftrightarrow (\frac{7}{3}, \frac{4}{3})$
$\begin{cases} x_4 = 0 \\ x_6 = 0 \end{cases} \Leftrightarrow (2, 2)$	$\begin{cases} x_5 = 0 \\ x_6 = 0 \end{cases} \Leftrightarrow (2, 1)$	$C_2^6 = \frac{6!}{(6-2)!\ 2!} = \frac{6 \times 5}{2} = 15$

The extreme points are the ones that are feasible.

The extreme points of the RM model

Solutions at which there are 2 ($= n - m$) binding constraints:

$$(0,0), (0,4), (0,6), (0,-1), (6,0), (3,0), (1,0),$$

$$(2,0), (\frac{3}{2},3), (3,2), (2,\frac{8}{3}), (\frac{7}{3},\frac{4}{3}), (2,2), (2,1).$$

The feasible solutions at which two constraints are binding, i.e., the extreme points, are:

$$(0,0), (0,4), (1,0), (\frac{3}{2},3), (2,2), (2,1),$$

and, the unfeasible solutions at which there are 2 binding constraints:

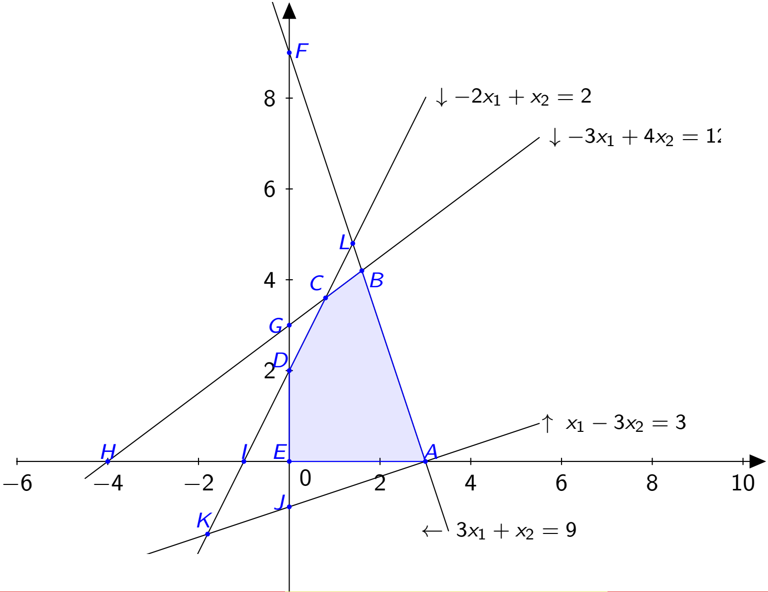
$$(0,6), (0,-1), (6,0), (3,0), (2,0), (3,2), (2,\frac{8}{3}), (\frac{7}{3},\frac{4}{3}).$$

Example 2:

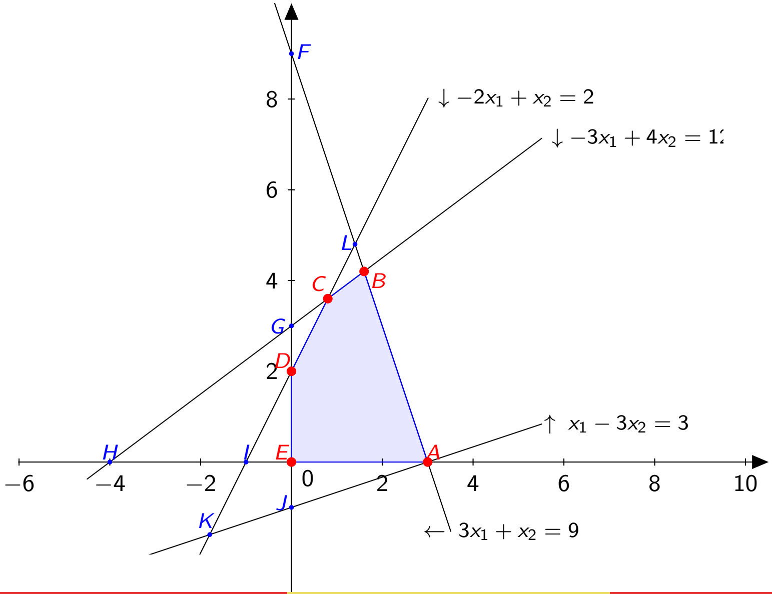
Identify the extreme points of the following LP.

$$
\begin{aligned}
\max\ & z = x_1 + 3x_2 \\
\text{s. to}\quad & x_1 - 3x_2 \leq 3 \\
& -2x_1 + x_2 \leq 2 \\
& -3x_1 + 4x_2 \leq 12 \\
& 3x_1 + x_2 \leq 9 \\
& x_1,\ x_2 \geq 0
\end{aligned}
$$

Example 2:


F
↓ $-2x_1 + x_2 = 2$
↓ $-3x_1 + 4x_2 = 12$
L
C
B
G
D
$\uparrow\ x_1 - 3x_2 = 3$
H
I
E
A
J
K
$\leftarrow\ 3x_1 + x_2 = 9$
8
6
4
2
−6
−4
−2
0
2
4
6
8
10

Example 2: extreme points are A, B, C, D, E



Solving with the Simplex method

Properties of an LP

P1 The Feasible Region of an LP problem is either an empty set or a convex set.

P2 If the Feasible Region of an LP problem is nonempty and bounded then at least an optimal solution exists.

P3 If an LP problem has optimum, then at least one of its extreme point feasible solution is an optimal solution.

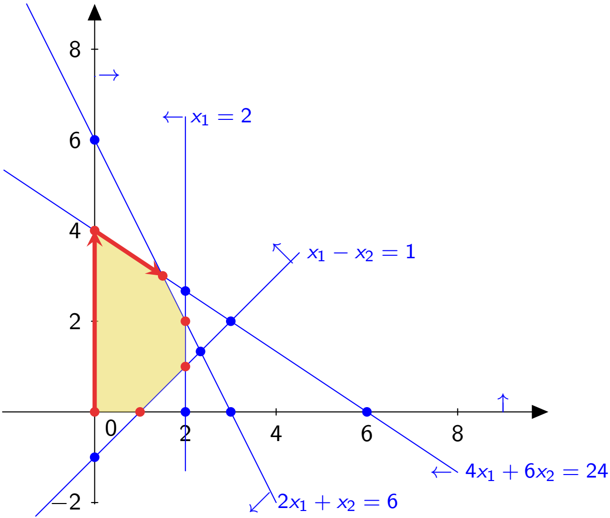
P4 Given an LP Problem with optimum, if an extreme point feasible solution has no adjacent extreme point feasible solution with a better value for the Objective Function then that point is an optimal solution.

The Simplex method

The simplex algorithm examines the extreme points until it finds the one that optimises the total value of the objective function or until it determines that the optimal solution occurs along an extreme direction

1. Consider an initial feasible solution corresponding to an extreme point: identify its non-binding constraints (basic variables) and build the Simplex table
2. Evaluate its adjacent extreme points and determine if there is one with a better objective function value.
3. If there is none: STOP the current extreme point is an optimal solution
4. Otherwise, identify a basic variable and a non-basic variable to exchange: obtain the new extreme point and go to Step 2.

The path of the Simplex method



The Simplex table

$$\begin{array}{ll} \max & z = c^T x \\ \text{s. t.:} & Ax = b \\ & x \geq 0 \end{array}$$

- $x \in \mathbb{R}^n$, $A_{m\times n}$ such that $n > m$ and $rank(A, b) = rank(A) = m$
- $B_{m\times m}$ invertible submatrix of matrix A ($rank(B) = m$)

The Simplex table is always associated to an extreme point and is

$\bar{z}$ and basic var. x_B	x	RHS
x_B	$B^{-1}A$	$B^{-1}b$
$\bar{z}$	$c_B B^{-1}A - c$	$c_B B^{-1}b$

The Simplex method: initial table

Obtain an initial extreme point (basic solution) and build the Simplex table

Use as an initial feasible basic solution (extreme point) the one obtained by setting the decision variables to zero: $x = (0, 0, 24, 6, 1, 2)$, with value $z = 0$, at which corresponds the identity basis $B = \mathcal{I}$

$\bar{z}$ and basic var	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_3	4	6	1	0	0	0	24
x_4	2	1	0	1	0	0	6
x_5	1	-1	0	0	1	0	1
x_6	1	0	0	0	0	1	2
$\bar{z}$	-4	-5	0	0	0	0	0

is the optimality condition satisfied?

Evaluate the current solution:if all values in the raw $\bar{z}$ satisfy the optimality condition, this is the optimal solution:

for a maximization problem, the optimality condition is $\bar{z} \geq 0$

for a minimization problem, the optimality condition is $\bar{z} \leq 0$

Otherwise, there are adjacent extreme points with a better objective function value

$\bar{z}$ and basic var	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_3	4	6	1	0	0	0	24
x_4	2	1	0	1	0	0	6
x_5	1	-1	0	0	1	0	1
x_6	1	0	0	0	0	1	2
$\bar{z}$	-4	-5	0	0	0	0	0

In this example, the current solution does not satisfy the optimality condition.

identify an entering variable in the basis

- select, from the raw $\bar{z}$, the variable/column corresponding to one that does not satisfy the optimality condition (for max problem, the most negative):
  select column of x_2, thus $j = 2$

$\bar{z}$ and basic var	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_3	4	6	1	0	0	0	24
x_4	2	1	0	1	0	0	6
x_5	1	-1	0	0	1	0	1
x_6	1	0	0	0	0	1	2
$\bar{z}$	-4	-5	0	0	0	0	0

identify a leaving variable from the basis

- select, from the column associated to variable x_j, in this example column of x_2, the variable that is leaving the basis:
obtain $\min\{\frac{b_i}{a_{ij}} \ : a_{ij} > 0\}$

$\bar{z}$ and basic var	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_3	4	6	1	0	0	0	24
x_4	2	1	0	1	0	0	6
x_5	1	-1	0	0	1	0	1
x_6	1	0	0	0	0	1	2
$\bar{z}$	-4	-5	0	0	0	0	0

in this example $\min\{\frac{24}{6}, \frac{6}{1}\} = \min\{4, 6\} = 4$ associate to the raw of x_3
thus, basic variable x_3 will leave the basis

The Simplex method

Use elementary operations on the matrix rows (Gauss elimination method for linear systems) to update the Simplex table and replace variable x_3 with x_2

$\bar{z}$ and basic var	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_3	4	6	1	0	0	0	24
x_4	2	1	0	1	0	0	6
x_5	1	-1	0	0	1	0	1
x_6	1	0	0	0	0	1	2
$\bar{z}$	-4	-5	0	0	0	0	0

The circled number (in the raw of x_3 and the column of x_3) is the pivot
The pivot raw is divided by the pivot
All the other elements in that column will turn to zero

The Simplex method

Obtain the following table

$\bar{z}$ and basic var	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_2	$\frac{2}{3}$	1	$\frac{1}{6}$	0	0	0	4
x_4	$4/3$	0	$-\frac{1}{6}$	1	0	0	2
x_5	$\frac{5}{3}$	0	$\frac{1}{6}$	0	1	0	5
x_6	1	0	0	0	0	1	2
$\bar{z}$	$-\frac{2}{3}$	0	$\frac{5}{6}$	0	0	0	20

This table corresponds to the extreme point / basic solution $x = (0, 4, 0, 2, 5, 2)$ with value $z = 20$

Evaluate this solution and repeat!

The Simplex method: optimal table

x_1 enters the base and x_4 leaves the base
Obtain the table

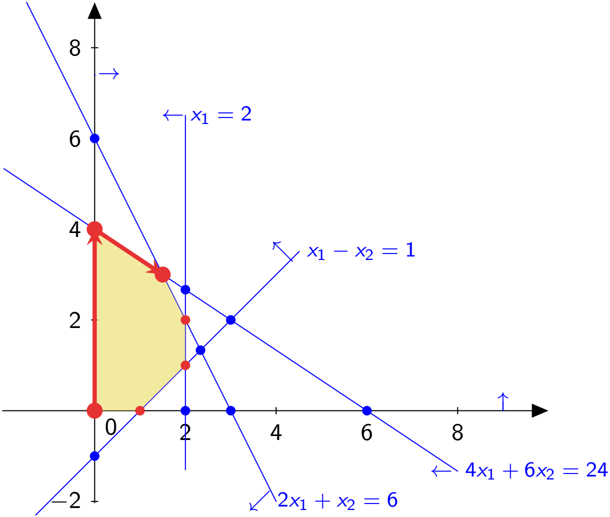
$\bar{z}$ and basic var	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_2	0	1	$\frac{1}{12}$	$-\frac{1}{2}$	0	0	3
x_1	1	0	$-\frac{1}{8}$	$\frac{3}{4}$	0	0	$\frac{3}{2}$
x_5	0	0	$\frac{3}{8}$	$-\frac{5}{4}$	1	0	$\frac{5}{2}$
x_6	0	0	$\frac{1}{8}$	$-\frac{3}{4}$	0	1	$\frac{1}{2}$
$\bar{z}$	0	0	$\frac{3}{4}$	$\frac{1}{2}$	0	0	21

This table corresponds to the extreme point / basic solution
$x = (\frac{3}{2}, 3, 0, 0, \frac{5}{2}, \frac{1}{2})$ with value $z = 21$

Evaluate this solution! This is the optimal solution.

Path of the Simplex method

the Simplex method evaluates the extreme points $(0,0)$; $(0,4)$; $(1.5,3)$;



Example: unique optimal solution

$$
\begin{array}{ll}
\max & z = -2x_1 + 4x_2 - 6x_3 \\
\text{s. t.:} & 3x_1 - 2x_2 - 4x_3 \leq 4 \\
& 2x_1 + x_2 + x_3 \leq 10 \\
& x_1 + 3x_2 - 2x_3 \leq 5 \\
& x_1, x_2, x_3 \geq 0
\end{array}
$$

x_B	x_1	x_2	x_3	x_4	x_5	x_6	$\bar{b}$
x_4	$11/3$	0	$-16/3$	1	0	$2/3$	$22/3$
x_5	$5/3$	0	$5/3$	0	1	$-1/3$	$25/3$
x_2	$1/3$	1	$-2/3$	0	0	$1/3$	$5/3$
$z_j - c_j$	$10/3$	0	$10/3$	0	0	$4/3$	$20/3$

Unique optimal solution!

$$x^* = (0, 5/3, 0, 22/3, 25/3, 0), \; z^* = 20/3$$

Example: alternative optimal solutions

Case of convex combination of two extreme points

$$\begin{array}{ll} \max & z = x_1 - 2x_2 + x_3 \\ \text{s. t.:} & x_1 + 2x_2 + x_3 \leq 12 \\ & 2x_1 + x_2 - x_3 \leq 6 \\ & -x_1 + 3x_2 \leq 9 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

after some iterations we obtain the following optimal table

x_B	x_1	x_2	x_3	x_4	x_5	x_6	$\overline{b}$
x_3	0	1	1	2/3	−1/3	0	6
x_1	1	1	0	1/3	1/3	0	6
x_6	0	4	0	1/3	1/3	1	15
$z_j - c_j$	0	4	0	1	0	0	12

$$x^{1*} = (6, 0, 6, 0, 0, 15),\ z^* = 12$$

however, there is an indication of an alternative optimum

Example: alternative optimal solutions

one more iteration

x_B	x_1	x_2	x_3	x_4	x_5	x_6	$\bar{b}$
x_3	1	2	1	1	0	0	12
x_5	3	3	0	1	1	0	18
x_6	-1	3	0	0	0	1	9
$z_j - c_j$	0	4	0	1	0	0	12

$$x^{2*} = (0, 0, 12, 0, 18, 9),\ z^* = 12$$

$$x^* = \alpha(6, 0, 6, 0, 0, 15) + (1 - \alpha)(0, 0, 12, 0, 18, 9), \alpha \in [0, 1],\ z^* = 12$$

Example: alternative optimal solutions

Case of optimal solutions along an extreme direction

$$\begin{array}{ll} \max & z = -4x_1 + 10x_2 \\ \text{s. t.:} & -3x_1 + 2x_2 \leq 3 \\ & -2x_1 + 5x_2 \leq 20 \\ & x_1, x_2 \geq 0 \end{array}$$

after some iterations we obtain the optimal table

x_B	x_1	x_2	x_3	x_4	$\overline{b}$
x_2	0	1	$-4/22$	$3/11$	$108/22$
x_1	1	0	$-5/11$	$2/11$	$25/11$
$z_j - c_j$	0	0	0	2	40

$$\overline{x}^* = (25/11, 108/22, 0, 0), \; z^* = 40$$

however, there is an indication of an alternative optimum, but this time it's along the direction

$$d^* = (5/11, 4/22, 1, 0)$$

$$x^* = (25/11, 108/22, 0, 0) + \beta(5/11, 4/22, 1, 0), \beta \geq 0, \; z^* = 40$$

Example: unbounded solution

$$\begin{array}{ll} \max & z = 2x_1 + 3x_2 \\ \text{s. t.:} & 2x_1 + 2x_2 \geq 6 \\ & -x_1 + x_2 \leq 1 \\ & x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{array}$$

after some iterations we obtain the table

x_B	x_1	x_2	x_3	x_4	x_5	$\bar{b}$
x_1	1	0	0	−1	1	2
x_2	0	1	0	0	1	3
x_3	0	0	1	−2	4	4
$z_j - c_j$	0	0	0	−2	5	13

problem unbounded along the direction

$$d^* = (1, 0, 2, 1, 0)$$

Decision Making and Optimization

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