1 Modeling and Solving Linear Programming Problems

1. For each one of the following problems write a linear programming model and solve it using the solver of the Excel.

   (a) Geppetto's Woodcarving, Inc., manufactures two types of wooden toys: soldiers and trains. Each soldier sells for \$27 and requires \$10 worth of raw materials. In addition, producing one soldier incurs \$14 in variable labor and overhead costs. Each train sells for \$21 and requires \$9 worth of raw materials, while producing one train incurs \$10 in variable labor and overhead costs. The manufacture of wooden soldiers and trains requires two types of skilled labor: carpentry and finishing. Each soldier requires 1 hour of carpentry and 2 hours of finishing labor, whereas each train requires 1 hour of carpentry and 1 hour of finishing labor. Each week, Geppetto can obtain all the raw materials needed for production but has only 80 hours of carpentry capacity and 100 hours of finishing capacity. The demand for trains is unlimited, whereas at most 40 soldiers can be sold per week. Geppetto wants to determine the weekly production quantities that maximize profit, defined as total revenue minus total production costs (revenues – costs).

   (b) Farmer Giles must determine how many acres of corn and wheat to plant this year. Each acre of corn produces 10 bushels and requires 4 hours of labor per week. Each acre of wheat produces 25 bushels and requires 10 hours of labor per week. Corn can be sold for \$3 per bushel, while wheat can be sold for \$4 per bushel. Farmer Giles has 7 acres of land and 40 hours of labor available each week. In addition, government regulations require him to produce at least 30 bushels of corn during the year.

      i. Let x_1 be number of acres of corn planted, and x_2 be number of acres of wheat planted. Using these decision variables, formulate an LP whose solution will tell Farmer Giles how to maximize the total revenue from wheat and corn.

      ii. Check the feasibility of the following points
      $(x_1 = 2, x_2 = 3)$
      $(x_1 = 4, x_2 = 3)$
      $(x_1 = 2, x_2 = 1)$
      $(x_1 = 3, x_2 = 2)$

      iii. Using the variables x_1 = number of bushels of corn produced and x_2 = number of bushels of wheat produced reformulate Farmer Giles's LP.

   (c) Truckco manufactures two types of trucks: 1 and 2. Each truck must go through the painting shop and assembly shop. If the painting shop were completely devoted to painting Type 1 trucks, then 800 per day could be painted; if the painting shop were completely devoted to painting Type 2 trucks, then 700 per day could be painted. If the assembly shop were completely devoted to assembling truck 1 engines, then 1,500 per day could be assembled; if the assembly shop were completely devoted to assembling truck 2 engines, then 1,200 per day could be assembled. Each Type 1 truck contributes \$300 to profit; each Type 2 truck contributes \$500. The profit of the company should be maximized.

(d) Al Ferris has \$60,000 that he wishes to invest now in order to use the accumulation for purchasing a retirement annuity in 5 years. After consulting with his financial adviser, he has been offered four types of fixed-income investments, which we will label as investments A, B, C, D. Investments A and B are available at the beginning of each of the next 5 years (call them years 1 to 5). Each dollar invested in A at the beginning of a year returns \$1.40 (a profit of \$0.40) 2 years later (in time for immediate reinvestment). Each dollar invested in B at the beginning of a year returns \$1.70 three years later. Investments C and D will each be available at one time in the future. Each dollar invested in C at the beginning of year 2 returns \$1.90 at the end of year 5. Each dollar invested in D at the beginning of year 5 returns \$1.30 at the end of year 5.

(e) A cargo plane has three compartments for storing cargo: front, center, and back. These compartments have capacity limits on both weight and space, as summarized below:

Compartment	Weight capacity (tons)	Space capacity (cubic feet)
Front	12	7000
Center	18	9000
Back	10	5000

Furthermore, the weight of the cargo in the respective compartments must be the same proportion of that compartment's weight capacity to maintain the balance of the airplane. The following quantities of four cargoes are available for shipment on an upcoming flight and have volume and profit as displayed:

Cargo	total Weight (tons) available	Volume (cubic feet/ton)	Profit (\$/ton)
1	20	500	320
2	16	700	400
3	25	600	360
4	13	400	290

Any portion of these cargoes can be accepted. The objective is to determine how much tons (if any) of each cargo should be accepted and how to distribute each among the compartments to maximize the total profit for the flight.

(f) A company has at its service 1000 skilled workers, semi-skilled 2300 workers and 800 unskilled workers. The semi-skilled workers can become skilled workers if they attend a one-year course, which costs 500 monetary units (m.u.) per worker. Unskilled workers can improve to be semi-skilled by training courses with a cost of 350 m.u. per worker. The company intends to plan the training of its staff over the next two years so that at the end of the planning period:

- unskilled workers cannot represent more than 10% of the total;
- at least 40% of the workers should attend a training course;
- at least 35% of the amount spent on training should be for unskilled workers

How much should the company spend on training courses?

(g) A farmer is raising pigs for market, and he wishes to determine the quantities of the available types of feed that should be given to each pig to meet certain nutritional requirements at a minimum total cost. The number of units of each type of basic nutritional ingredient contained within a kilogram of each feed type is given in the following table, along with the daily nutritional requirements and feed costs.

Nutritional Ingredient	Kg of Corn	Kg of Tankage	Kg of Alfalfa	Min Daily Requirement
Carbohydrates	90	20	40	200
Protein	30	80	60	180
Vitamins	10	20	60	150
Cost (m.u.)	42	36	30	

The objective is to minimize the cost of feeding one pig.

(h) A plant imports three types of thread - cotton, wool and fiber - to produce three different types of cloth - C1, C2 and C3. The cloths should follow the specifications below:

Cloth	Selling price (m.u./kg)	Specifications
C1	680	At least 60% of cotton and at most 20% of fiber
C2	570	At most 60% of fiber and at least 15% of wool
C3	450	At most 50% of fiber

The aim is to determine the production plan so that the profit is maximized, using the information about availabilities and cost given in the following table:

Thread	Availabilities (kg)	Cost (m.u./kg)
Cotton	2 000	700
Wool	2 500	500
Fiber	1 200	400

2. **Portfolio Optimization under Risk Constraints.** A company is given three financial assets for investment. The goal is to allocate a total investment budget across these assets in order to maximize the portfolio's expected return, subject to risk and diversification constraints.

The expected returns and linearized risk contributions for each asset are:

Asset i	Expected Return μ_i	Risk Contribution ρ_i	Bounds $[l_i, u_i]$
1	8.0%	0.05	[0, 0.7]
2	6.0%	0.02	[0, 0.7]
3	10.0%	0.10	[0, 0.7]

The total portfolio must satisfy:

- all capital must be invested (i.e., the weights must sum to 1),
- the total linear risk exposure must not exceed $R^{\max} = 0.06$.

(a) Formulate the problem as a linear program.

(b) Solve the LP using a software tool of your choice (e.g., Python + `scipy.optimize.linprog`, Python + PuLP, Excel Solver, XPRESS, Gurobi, etc.).

(c) Report the optimal solution, the expected return and risk of the optimal portfolio.

(d) Interpret the results: which asset is most preferred? How tight is the risk constraint?

(e) Try slightly increasing $R^{\max}$ (e.g., to 0.065). How does the optimal solution change? What does this tell you about the sensitivity of the portfolio to risk appetite?

3. Solve the following problems using the graphical method, the Solver of Excel and the Simplex method.

(a) $\max\ z = x_1 + 2x_2$
s. a
$$\begin{aligned} x_1 - 2x_2 &\leq 3 \\ x_1 + x_2 &\leq 3 \\ x_1, x_2 &\geq 0 \end{aligned}$$

(b) $\max\ z = 3x_1 + 4x_2$
s. a
$$\begin{aligned} x_1 - 2x_2 &\geq 4 \\ x_1 + x_2 &\leq 3 \\ x_1, x_2 &\geq 0 \end{aligned}$$

(c) $\max\ z = x_1 + x_2$
s. a
$$\begin{aligned} x_1 - x_2 &\leq 2 \\ x_1 - x_2 &\geq 0 \\ x_1, x_2 &\geq 0 \end{aligned}$$

(d) $\max\ z = x_1 - x_2$
s. a
$$\begin{aligned} x_1 - x_2 &\leq 2 \\ x_1 - x_2 &\geq 0 \\ x_1, x_2 &\geq 0 \end{aligned}$$

(e) $\max\ z = 3x_1 + 6x_2$
s. a
$$\begin{aligned} x_1 + 2x_2 &\leq 4 \\ x_1 - x_2 &\geq 0 \\ x_1, x_2 &\geq 0 \end{aligned}$$

(f) $\min\ z = x_1 + x_2$
s. a
$$\begin{aligned} x_1 - x_2 &\leq 2 \\ x_1 - x_2 &\geq -2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

(g) $\max\ z = -10x_1 - 5x_2$
s. a
$$\begin{aligned} x_1 - x_2 &\leq 5 \\ x_1 + \tfrac{8}{5}x_2 &\geq -3 \\ x_1 \in \mathbb{R},\ x_2 &\geq 0 \end{aligned}$$

(h) $\min\ z = x_1 + x_2$
s. a
$$\begin{aligned} 2x_1 + x_2 &\geq 4 \\ x_1 - x_2 &\leq 1 \\ x_1, x_2 &\geq 0 \end{aligned}$$

4. A company produces two products, Product A and Product B. The production of each product requires time on two machines, Machine 1 and Machine 2. The company has a total of 100 hours of time available on Machine 1 and 200 hours of time available on Machine 2. The production of Product A requires 2 hours on Machine 1 and 4 hours on Machine 2. The production of Product B requires 3 hours on Machine 1 and 1 hour on Machine 2. The profit per unit of Product A is 10 and the profit per unit of Product B is 15. The company wants to maximize its profit.

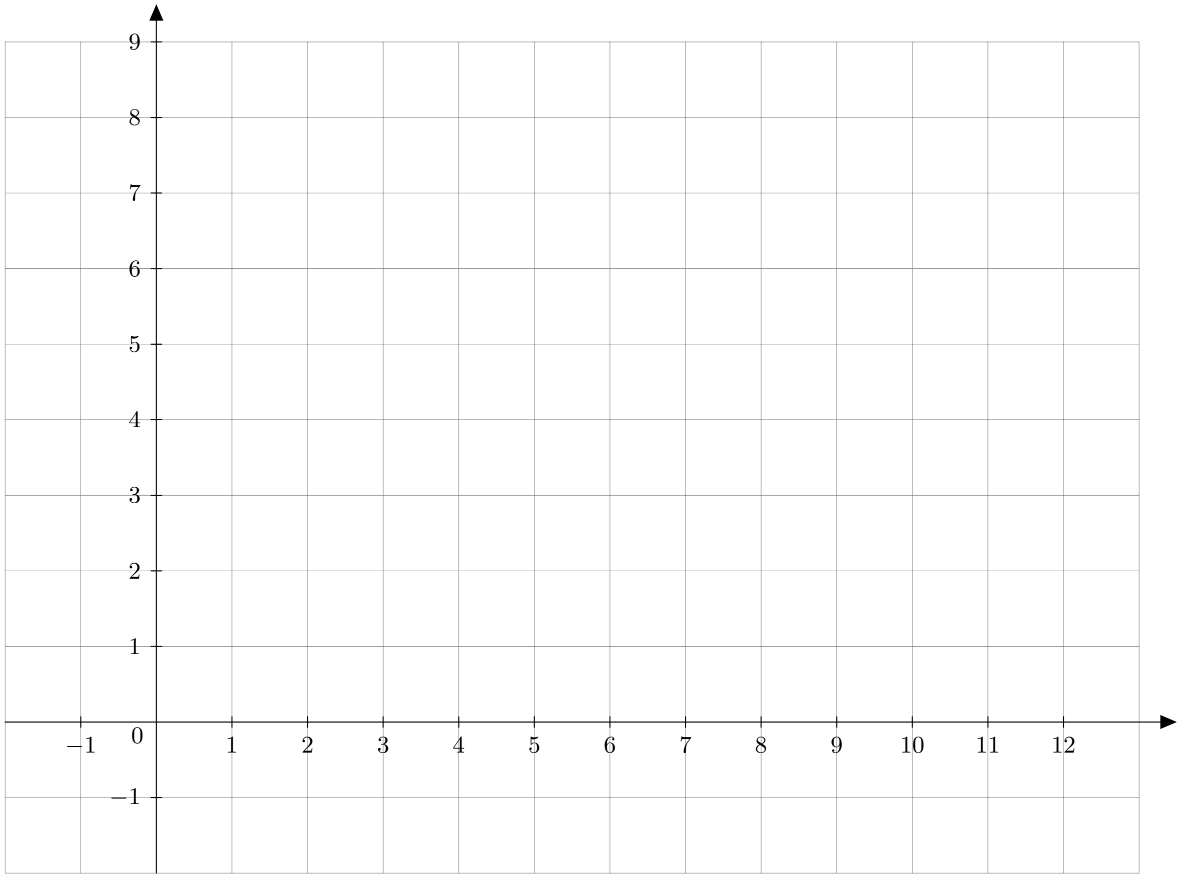
(a) Propose a linear programming formulation to this problem.

(b) Use the graphical method to solve the problem.

(c) Interpret the solution obtained.

5. Use the graphical method to solve the following LP problem and answer the questions that follow.

$$
\begin{aligned}
\max \quad & 3x_1 + 5x_2 \\
\text{s.t.} \quad & x_1 + 2x_2 \geq 4, \\
& x_1 + 3x_2 \leq 12, \\
& 3x_1 - x_2 \leq 6, \\
& x_1, x_2 \geq 0.
\end{aligned}
$$



(a) How many extreme points are there? What are the values of x_1 and x_2 at each one?

(b) Take four points at your choice so that you can match the corresponding solutions to the following categories:

a feasible basic solution: ____________ an infeasible non-basic solution: ____________

a feasible non-basic solution: ____________ an infeasible basic solution: ____________

(c) If the criteria is changed to minimization and the objective function is changed to $4x_1 + 2x_2$, what will the optimal solution be?

Some solutions

1.1a $x^* = (20, 60),\ z^* = 180,$

x_1 number of soldiers to produce per week

x_2 number of trains to produce per week

$$\begin{aligned} \max \quad & z = 3x_1 + 2x_2, \\ \text{s. to:} \quad & 2x_1 + x_2 \le 100, \\ & x_1 + x_2 \le 80, \\ & x_1 \qquad \le 40, \\ & x_1, x_2 \ge 0. \end{aligned}$$

1.1b (i) $x^* = (3, 2.8),\ z^* = 370$. (iii) $x^* = (30, 70),\ z^* = 370$.

1.1c $x^* = (0, 700),\ z^* = 350000,$

x_j number of trucks of type j to produce per day, $j = 1, 2,$

$$\begin{aligned} \max \quad & z = 300x_1 + 500x_2, \\ \text{s. to:} \quad & \frac{1}{800}x_1 + \frac{1}{700}x_2 \le 1, \\ & \frac{1}{1500}x_1 + \frac{1}{1200}x_2 \le 1, \\ & x_1, x_2 \ge 0. \end{aligned}$$

1.1d On year 1 invest 60000 on A, on year 3 invest 84000 on A, and on year 5 invest 117600 on D; $x^* = (60000, 0, 0, 0, 0, 84000, 0, 0, 117600),\ z^* = 152880.$

1.1e

1.1f 840 semi-skilled workers will become skilled workers, and 800 unskilled workers will become semi-skilled workers, $x^* = (840, 800, 0),\ z^* = 700000.$

1.1g $x^* = (1.143; 0; 2.429),\ z^* = 120.857.$

1.1h $x^* = (2000; 0; 0; 666.667; 1833.333; 0; 666.667; 533.333; 0),\ z^* = 485666.667.$

1.2 Data:

μ_i	expected return of asset i, $i = 1, \ldots, n$
l_i, u_i	minimum and maximum portfolio weights for asset i
ρ_i	linearized measure of risk exposure for asset i
$R^{\max}$	maximum total risk exposure allowed

Decision variables:

$$w_i \ge 0 \quad \text{portfolio weight assigned to asset } i,\ i = 1, \ldots, n.$$

Formulation:

$$
\begin{aligned}
\max_{w} \quad & \sum_{i=1}^{n} \mu_i w_i && \text{(maximize expected return)} \\
\text{s.t.} \quad & \sum_{i=1}^{n} w_i = 1 && \text{(capital fully invested)} \\
& l_i \le w_i \le u_i && \forall i \quad \text{(weight bounds)} \\
& \sum_{i=1}^{n} \rho_i w_i \le R^{\max} && \text{(risk constraint)} \\
& w_i \ge 0 && \forall i
\end{aligned}
$$

Notes:

- This is a linearized version of the Markowitz model.
- Variance or VaR constraints are approximated by linear inequalities.

Optimal solution: $x^* = (0.7000,\ 0.0625,\ 0.2375)$.

Expected return: 8.35%.

Total risk exposure: 0.0600.

1.3.(a) unique optimal solution $x^* = (0, 3),\ z^* = 6$.

1.3.(b) empty feasible region, problem impossible.

1.3.(c) unbounded feasible region, problem unbounded.

1.3.(d) alternative optimal solutions, semi-line $x^* = (2.0) + \beta(1, 1), \beta \ge 0,\ z^* = 2$. The extreme point $(2, 0)$ is an optimal solution and all the points in the semi-line $(2, 0) + \beta(1, 1)$ are also optimal.

1.3.(e) alternative optimal solutions, the segment line between the points $(4, 0)$ and $(\frac{4}{3}, \frac{4}{3}),\ z^* = 12$. We can write the segment line as $x^* = \alpha(\frac{4}{3}, \frac{4}{3}) + (1 - \alpha)(4, 0), \alpha \in [0, 1]$ and, alternatively, we can also write $x^* = \alpha_1(4, 0) + \alpha_2(\frac{4}{3}, \frac{4}{3}),\ \alpha_1 + \alpha_2 = 1, \alpha_1, \alpha_2 \ge 0$.

1.3.(f) unique optimal solution, $x^* = (0, 0),\ z^* = 0$.

1.3.(g) unique optimal solution, $x^* = (-3, 0),\ z^* = 30$.

1.3.(h) unique optimal solution, $x^* = (\frac{5}{3}, \frac{2}{3}),\ z^* = \frac{7}{3}$.

1.4. (a) Decision variables: let x be the number of units of Product A produced and y be the number of units of Product B produced.

The objective function is to maximize the profit: $Z = 10x + 15y$.

The constraints are:

time constraint on Machine 1: $2x + 3y \le 100$,

time constraint on Machine 2: $4x + y \le 200$,

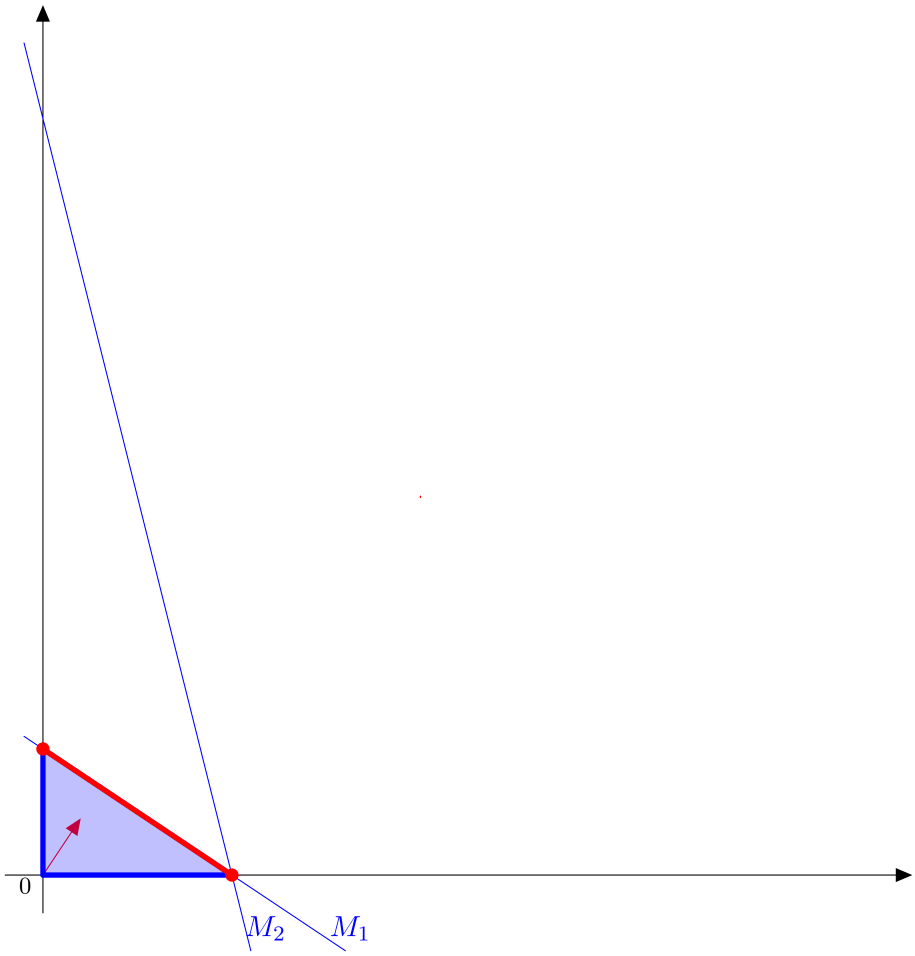
non-negativity constraints: $x \ge 0, y \ge 0$.

The complete model is

$$\begin{aligned} \max \quad & Z = 10x + 15y \\ \text{s.t.} \quad & 2x + 3y \leq 100, \\ & 4x + y \leq 200, \\ & x \geq 0, y \geq 0. \end{aligned}$$

The first constraint states that the total time spent on Machine 1 cannot exceed 100 hours. The second constraint states that the total time spent on Machine 2 cannot exceed 200 hours. The non-negativity constraint states that the number of units produced of each product cannot be negative.

(b) The problem has several optimal alternative solutions $x^* = \alpha(0,\ 33 + \frac{1}{3}) + (1-\alpha)(50, 0) = \alpha(0, \frac{100}{3}) + (1-\alpha)(50, 0)$, with $\alpha \in [0, 1]$, all with value $z = 500$. See the graphical resolution sketch below.



(c) To achieve a maximum profit of $z = 500$, there are several alternative solutions, the ones in the segment line displayed in red, between the two extreme points $(0, \frac{100}{3})$ and $(50, 0)$. Here are described the optimal solutions that correspond to extreme points.

One alternative optimal solution is $x^* = (0, 33 + \frac{1}{3}) = (0, \frac{100}{3})$ which means that the optimal plan of the company is to produce $\frac{100}{3}$ units of Product B and not produce Product A. This plan utilizes the full time of Machine 1, 100 hours, and $\frac{100}{3}$ hours of the 200 hours available from Machine 2.

Another alternative optimal solution is $x^* = (50, 0)$ which means that the optimal plan of the company is to produce 50 units of Product A and not produce Product B. This plan utilizes the full time of Machine 1, 100 hours, and the full time, 200 hours, of Machine 2.

1.5 (a) There are four extreme points: $(0, 2), (0, 4), (3, 3), (\frac{16}{7}, \frac{6}{7})$

(b) For example, a feasible basic solution: $(3, 3)$; an infeasible non-basic solution: $(5, 5)$; a feasible non-basic solution: $(1, 3)$; an infeasible basic solution: $(-12, 8)$

(c) optimal solution is $(0, 2)$ with value $z_{min} = 4$