2 Duality and Sensitivity Analysis

1. A cautious investor, after a difficult experience with real estate-based funds, has recovered 25000 monetary units (m.u.) and now wants to invest this amount over a certain period. This time, the investor's main goal is to **minimize risk**, while still achieving a minimum return of 2000 m.u. by the end of the investment period. After analyzing two available financial products, the investor builds the following linear programming model:

$$\begin{array}{lll} \min & z = x_1 + 2x_2 & \text{(overall risk index)} \\ \text{s. to} & x_1 + x_2 \leq 25 & \text{(budget constraint)} \\ & 0.5x_1 + 0.8x_2 \geq 2 & \text{(minimum return constraint)} \\ & x_1 \geq 0, \quad x_2 \geq 0 & \text{(non-negativity)} \end{array}$$

   where x_1 is the amount (in thousands of m.u.) to invest in Product 1, and x_2 is the amount (in thousands of m.u.) to invest in Product 2.

   (a) Graphically solve the linear programming problem. Present and interpret the optimal values of the decision variables and the slack/surplus variables.

   (b) Formulate the dual of the problem and determine its optimal solution (only the decision variables). You may use the primal solution and resolution to support your answer.

   (c) Determine the range of values for the right-hand side coefficients (budget and return) for which the current optimal basis remains unchanged.

   (d) Determine the allowable range for the objective function coefficients such that the current solution remains optimal.

2. In factory Choco, three new types of chocolate bars are going to be made for the food industry. Each bar is made of sugar and chocolate only. The table below shows the quantity of sugar and chocolate (in kg) and the profit (in monetary units) of each chocolate bar.

Bar	Sugar (kg/bar)	Chocolate (kg/bar)	Profit (m.u./bar)
Type 1	1	2	3
Type 2	1	3	7
Type 3	1	1	5

   The factory has 50 kg of sugar and 100 kg of chocolate available. To formulate the problem, we define variables x_j, representing the number of chocolate bars Type j to make, where $j = 1, 2, 3$. Answer to the following questions using, when needed, the Solver/Excel to find the solution of the LP problems.

   (a) For which unit profit values of chocolate bars of Type 2 does the current solution remains optimal? Which will be the optimal solution in case the unit profit is 13 m.u.?

   (b) Is it worth considering an increase in the availability of sugar?

(c) For which amount of sugar is the set of basic variables in the optimal solution the same?

(d) Is it worth considering an increase in the availability of chocolate?

(e) For which amount of chocolate is the set of basic variables in the optimal solution the same?

(f) If the amount of sugar available was of 60 kg, which would be the total profit of these products? Which should be the production plan that Choco should apply in these conditions?

(g) Repeat the previous question for an availability of sugar of 40 kg and 30 kg.

3. A humanitarian organization intends to plan a program for medicines distribution in two regions located in the Great Lakes area of Africa. For strategic and security purposes it is possible to use 3 airports from which, by land routes, the supply of the two regions will take place. Considering that the transportation cost of medicaments to the airports should be minimized, insuring, in each of the two regions, a minimum number of people is contemplated by the program, the following LP model has been formulated:

$$\begin{aligned} \min z &= 40x_1 + 18x_2 + 30x_3 \quad \text{(in m.u.)} \\ \text{s.t. } & 4x_1 + x_2 + x_3 \geq 250 \quad \text{(thousands of people)} \\ & 4x_1 + 3x_2 + 6x_3 \geq 350 \quad \text{(thousands of people)} \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

where x_j represents the tons of medicaments to be shipped to airport $j = 1, 2, 3$.

(a) Obtain the optimal solution for the problem using the Solver/Excel.

(b) Write a short report presenting the problem's solution, referring the value of the dual decision variables as well as its meaning.

(c) Consider that the number of thousands of people to be contemplated in the first region is 350 instead of 250, which will be the new cost for the program?

(d) Determine the changes in the solution if airport 1 cannot receive more than 40 ton. of medicaments.

4. A non-profit organization collects donations of medicine (M), food (F), and clothing (C). In a few days, there will be transportation to an institution, and there is a need to organize the loading. The selection team, whether selecting only medicines, clothes or food, has the capacity to process 20 tonnes. Selecting one tonne of medication takes three times longer than selecting one tonne of clothes or food. The packaging team can process 10 tonnes of donations. Packaging medication and food takes three times and twice as long as packaging clothes (in tonnes), respectively. This shipment must contain at least 2 tonnes of medication. The following LP problem was formulated to decide which goods to prepare for the next shipment:

$$\begin{array}{llllllll} \max & Z = 2x_M + 2x_F + x_C & & & & & & \\ \text{s.t.} & 3x_M & + & x_F & + & x_C & \leq & 20 \\ & 3x_M & + & 2x_F & + & x_C & \leq & 10 \\ & x_M & & & & & \geq & 2 \\ & x_M, & & x_F, & & x_C & \geq 0 & \end{array}$$

Where x_j represents the tons of donation of type j ($j = M, F, C$) to prepare for shipment. The objective function translates the total utility considering that one ton of medicine has the same utility as one ton of food, which in turn is twice that of clothes. If necessary, refer to the Solver/Excel report available next to answer the questions.

Microsoft Excel 16.0 Sensitivity Report

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
\$B\$9	variable xM	2	0	2	1	1E+30
\$C\$9	variable xF	2	0	2	1E+30	0
\$D\$9	variable xC	0	0	1	0	1E+30

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
\$E\$5	selection capacity	8	0	20	1E+30	12
\$E\$6	packaging capacity	10	1	10	24	4
\$E\$7	minimum meds	2	-1	2	1.333333333	2

(a) Write and interpret the solution of the problem, including the value of the objective function and the slack (or auxiliary) variables.

(b) Write the dual.

(c) Interpret the meaning of the first dual variable.

(d) Determine the optimal solution of the dual (decision and slack variables) by complementarity relations.

Note: If you have not solved (4d) and need these values in the following questions, assume $y_1 = 0, y_2 = 2, y_3 = -2$.

(e) Determine the change in total utility if it were not mandatory to transport medications.

(f) What would be the new optimal solution and optimal value if the utility of a ton of medicines decreased to 1 per ton?

5. Write the dual problem and the complementarity primal/dual relations of the following LP problem.

$$\begin{aligned} \max \quad & 3x_1 + 5x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \geq 4, \\ & x_1 + 3x_2 \leq 12, \\ & 3x_1 - x_2 \leq 6, \\ & x_1, x_2 \geq 0. \end{aligned}$$

6. C Sports produces three different types of baseball gloves: a low-cost model, a regular model and a catcher's model. The company has 900 hours of production time available in its cutting and sewing department, 300 hours available in its finishing department and 100 hours available in its packing and shipping department. Given the production time requirements and the profit contribution per glove, the following LP with decision variables x_1, x_2 and x_3 has been solved to maximize the total profit contribution.

$$
\begin{aligned}
(P)\ \max\quad & 3x_1 + 5x_2 + 8x_3 \\
\text{s.t.}\quad & \frac{1}{2}x_1 + x_2 + \frac{3}{2}x_3 \leq 900 \\
& \frac{1}{6}x_1 + \frac{1}{2}x_2 + \frac{1}{3}x_3 \leq 300 \\
& \frac{1}{8}x_1 + \frac{1}{8}x_2 + \frac{1}{4}x_3 \leq 100 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

The sensitivity report obtained using Excel Solver is shown next.

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
\$B\$2	x1	0	-1	3	1	1E+30
\$C\$2	x2	500	0	5	7	1
\$D\$2	x3	150	0	8	2	2

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
\$E\$5	r1	725	0	900	1E+30	175
\$E\$6	r2	300	3	300	100	166,667
\$E\$7	r3	100	28	100	35	25

(a) How many gloves of each model should C Sports manufacture? Write and interpret the optimal production plan. Include the value of the slack variables.

(b) What is the total profit contribution C Sports can earn with the given production quantities?

(c) How many hours of production time will be scheduled in each department?

(d) What are the dual values for the resources? Interpret each.

(e) If overtime can be scheduled in one of the departments, where would you recommend doing so? Why?

(f) What profit contribution for low-cost gloves will lead to its production?

(g) What is the maximum amount that C Sports should agree to pay to have more hours available in its finishing department, and continue with the same production plan. How many additional hours can be considered?

(h) How much will the value of the optimal solution improve if 20 extra hours of packaging and shipping time are made available?

Some solutions

2.1 $x^* = (4,0),\ z^* = 4 = w^*, y^* = (0,2)$.

2.2 $x^* = (0,25,25),\ z^* = 300 = w^*, y^* = (4,1)$.

2.3 $x^* = (57.5;0;20),\ z^* = 2900 = w^*, y^* = (6,4)$.

2.4 (a) $x^* = (2,2,0,12,0,0)$ with value $z^* = 8$

(b)

$$\begin{aligned} \min \quad & W = 20y_1 + 10y_2 + 2y_3 \\ \text{s.t.} \quad & 3y_1 + 3y_2 + y_3 \geq 2, \\ & y_1 + 2y_2 \geq 2, \\ & y_1 + y_2 \geq 1, \\ & y_1, y_2 \geq 0, \\ & y_3 \leq 0. \end{aligned}$$

(c) The first dual variable y_1 is associated to the first constraint of the primal which is related with the capacity of the selection process.

It's optimal value is 12. It means that the associated primal constraint is binding (is satisfied as an equality). When the actual basis is mantained, it represents the amount of the increase value, in the objective function, when the RHS (right hand side) value $b_1 = 20$ (the capacity of the selection process) of the associated constraint increases one unit.

(d) The complementarity relations are

$$\begin{aligned} (20 - (3x_M + x_F + x_C))y_1 &= 0, \\ (10 - (3x_M + 2x_F + x_C))y_2 &= 0, \\ (x_M - 2)y_3 &= 0, \\ x_M(3y_1 + 3y_2 + y_3 - 2) &= 0, \\ x_F(y_1 + 2y_2 - 2) &= 0, \\ x_C(y_1 + y_2 - 1) &= 0. \end{aligned}$$

The primal solution is $x^* = (2,2,0,12,0,0)$ thus the previous relations become

$$\begin{aligned} 12y_1 &= 0, \\ 0y_2 &= 0, \\ 0y_3 &= 0, \\ 2(3y_1 + 3y_2 + y_3 - 2) &= 0, \\ 2(y_1 + 2y_2 - 2) &= 0, \\ 0(y_1 + y_2 - 1) &= 0. \end{aligned}$$

Hence

$$\begin{aligned}
&y_1 = 0,\\
&\text{nothing can be concluded about } y_2,\\
&\text{nothing can be concluded about } y_3,\\
&(3y_1 + 3y_2 + y_3 - 2) = 0,\\
&(y_1 + 2y_2 - 2) = 0,\\
&\text{nothing can be concluded.}
\end{aligned}$$

Therefore

$$\begin{aligned}
&y_1 = 0,\\
&3y_1 + 3y_2 + y_3 - 2 = 0,\\
&y_1 + 2y_2 - 2 = 0.
\end{aligned}$$

is equivalent to

$$\begin{aligned}
&y_1 = 0,\\
&3y - 2 + y_3 - 2 = 0,\\
&2y_2 - 2 = 0.
\end{aligned}$$

and gives

$$\begin{aligned}
&y_1 = 0,\\
&y_3 = -1,\\
&y_2 = 1.
\end{aligned}$$

The decision variables of the dual are $y^* = (0, 1, -1)$ and by replacing in the constraints we obtain the value of the slack variables $y_4 = y_5 = y_6 = 0$. We have $y^* = (0, 1, -1, 0, 0, 0)$.

(e) The variable y_3 is the dual variable associated with constraint 3. It has a value of $y_3 = -1$. This means that decreasing the RHS of this constraint by 1 will increase the total utility by 1, while maintaining the current basis. The Sensitivity Report shows that the allowable decrease of b_3 is 2. Therefore, decreasing b_3 will increase the objective function value. To achieve the greatest increase in the objective function value, we must increase b_3 by the maximum amount possible, which is 2. Each unit increase in b_3 results in a corresponding increase of 1 in the objective function, resulting in a total increase of 2 units. The total utility will increase by 2 units, resulting in a total of 10.

(f) A ton of medicines has a utility of $c_M = 2$. According to the Sensitivity Report, the allowable decrease is infinite. Therefore, the optimal solution remains unchanged, $x^* = (2, 2, 0, 12, 0, 0)$, while maintaining the current basis, but the value of the objective function decreases by 2 for each unit the utility decreases, resulting in a new value of 6.

2.5 dual:

$$\begin{aligned} \min \quad & 4y_1 + 12y_2 + 6y_3 \\ \text{s.t.} \quad & y_1 + y_2 + 3y_3 \geq 3, \\ & 2y_1 + 3y_2 - y_3 \geq 5, \\ & y_1 \leq 0,\; y_2, y_3 \geq 0. \end{aligned}$$

complementarity prima/ dual relations:

$$\begin{aligned} & x_1(y_1 + y_2 + 3y_3 - 3) = 0, \\ & x_2(2y_1 + 3y_2 - y_3 - 5) = 0, \\ & (x_1 + 2x_2 - 4)y_1 = 0, \\ & (12 - x_1 - 3x_2)y_2 = 0, \\ & (6 - 3x_1 + x_2)y_3 = 0. \end{aligned}$$

2.6 (a) $x^* = (0, 500, 150, 175, 0, 0)$

(b) $z^* = 3700$

(c) section 1: 725; section 2: 300; section 3: 100;

(d) $y^* = (0, 3, 28, 1, 0, 0)$ interpret the solution

(e) Section 3. Constraint 3 has the highest shadow price, equal to 28. Hence, a one-unit increase in its right-hand side (overtime limit) would raise the optimal objective value by 28, provided this change stays within the allowable range.

(f) The objective function coefficient corresponding to the low-cost gloves is c_1, and its allowable range is $(-\infty,\ 1]$, that is $\Delta c_1 \in (-\infty,\ 1]$, therefore its profit must satisfy $\bar{c}_1 > 3 + 1$.

(g) The shadow price for Section 2 is 3 per hour, Section 2 allowable increase is 100 hours, thus RHS can increase from 300 to $300 + 100 = 400$.

Therefore the maximum total amount C Sports should agree to pay (while the current plan stays optimal) is $3 \times 100 = 300$, and these results hold only within the allowable range.

(h) The shadow price for Section 3 is 28, and the allowable increase for this constraint is 35 hours. Since the proposed increase of 20 hours is within this allowable range ($20 < 35$), the improvement in the optimal objective value is $20 \times 28 = 560$.

Thus, the value of the optimal solution will increase by 560.