Decision Making and Optimization

Duality and Sensitivity Analysis

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Duality and Sensitivity Analysis

The Dual problem

Construction of the Dual problem

max			min
constraint i type	$\leq$ $\geq$ $=$	≥ 0 ≤ 0 $\in \mathbb{R}$	variable i
variable j	≥ 0 ≤ 0 $\in \mathbb{R}$	$\geq$ $\leq$ $=$	constraint j
matrix A			matrix A^T
o.f. coefic.			RHS
RHS			o.f. coefic.

Example

PRIMAL

$$
\begin{aligned}
\max \quad & z = 4x_1 + 5x_2, \\
\text{s. to:} \quad & 4x_1 + 6x_2 \leq 24, \\
& 2x_1 + x_2 \leq 6, \\
& x_1 - x_2 \leq 1, \\
& x_1 \qquad \leq 2, \\
& x_1, x_2 \geq 0,
\end{aligned}
$$

DUAL

$$
\begin{aligned}
\min \quad & w = 24y_1 + 6y_2 + y_3 + 2y_4, \\
\text{s. to:} \quad & 4y_1 + 2y_2 + y_3 + y_4 \geq 4, \\
& 6y_1 + y_2 - y_3 \qquad \geq 5, \\
& y_1, y_2, y_3, y_4 \geq 0,
\end{aligned}
$$

Example

PRIMAL

$$\begin{aligned} \max \quad & z = 4x_1 + 5x_2, \\ \text{s. to:} \quad & 4x_1 + 6x_2 \leq 24, \\ & 2x_1 + x_2 \leq 6, \\ & x_1 - x_2 \leq 1, \\ & x_1 \qquad \leq 2, \\ & x_1, x_2 \geq 0, \end{aligned}$$

DUAL

$$\begin{aligned} \min \quad & w = 24y_1 + 6y_2 + y_3 + 2y_4, \\ \text{s. to:} \quad & 4y_1 + 2y_2 + y_3 + y_4 \geq 4, \\ & 6y_1 + y_2 - y_3 \qquad \geq 5, \\ & y_1, y_2, y_3, y_4 \geq 0, \end{aligned}$$

Primal/Dual Relations

primal variables are connected with dual constraints

dual variables are connected with primal constraints

dual objective function coefficients are connected with the primal rhs

- The i-th shadow price (optimal value of dual variable i) represents the proportion of the change in the optimal value of the primal due to an increase in the i-th right-hand-side

The dual of the dual is the primal.

Fundamental properties

- The value of the o.f. for any feasible solution to the maximization problem is not greater than the value of the o.f. corresponding to a solution to the minimization problem
- If x and y are feasible solutions to a pair of primal and dual problems that give equal values to the respective o.f., then x and y are optimal solutions to the respective primal and dual problems
- For any pair of dual problems, the existence of finite optimal solutions for one of them guarantees the existence of finite optimal solutions for the other, and the respective optimal values of the o.f. coincide
- A LP problem has a finite optimal solution if and only if there are feasible solutions for the primal and the dual
- If for one of the problems there is an unbounded solution, then the other does not has feasible solutions

Obtain the Dual Optimal Solution

Obtain the Dual Optimal Solution

Knowing the Optimal Primal Solution

- Graphically: to find y_i solve LP with $\Delta b_i = +1$, then $y_i = \Delta z$
- Simplex optimal table: y_i are the values in the z row of the i-th constraint slack variable
- Sensitivity report of the Excel Solver: "Shadow Price" column.
- Complementarity primal/dual relations:
  If a constraint in (P) is not binding, then the dual variable is zero
  If a variable in (P) is non-zero, then the dual constraint is binding