Lecture 3

1. Summary of Lecture 1: Technology

1.1. Describing technologies

1.2. Properties of technologies

1.3. Properties of technologies cont.: TRS

2. Summary of Lecture 2: Profit maximization

2.1. The profit maximization problem:

- $\Pi(p) = \underset{y}{Max}\, py \; s.t.\; y \in Y$ or $\Pi(p) = \underset{y}{Max}\, py \; s.t.\; T(y) = 0$
- One output: $\Pi(p) = \underset{x}{Max}\, pf(x) - wx \; s.t.\; x \geq 0$

2.2. Implications of profit maximization

2.2.1. **Factor demand function** x(p,**w**) **and supply function** y(p,**w**)=f(x(p,**w**)).

2.2.2. Comparative statics: the Weak Axiom of Profit Maximization (WAPM) implies that $\triangle\mathbf{p}\triangle\mathbf{y} \geq 0$.

2.3. The profit function

2.3.1. Properties

- $\Pi(\cdot)$ is increasing (decreasing) in p_i if *i* is an output (input);
- $\Pi(\cdot)$ is homogeneous of degree 1 in $\boldsymbol{p}$;
- $\Pi(\cdot)$ is convex in $\boldsymbol{p}$; intuition: by responding to the price change and adjusting the output level or input mix, the firm is at least as well of as if it did not respond;
- $\Pi(\cdot)$ is continuous in $\boldsymbol{p}$;
- *Hotelling's lemma*: $\frac{\partial \Pi(p)}{\partial p_i} = y_i(p)$, for all i=1,…,n, assuming that the derivative exists and $p_i > 0$.

2.3.2. Implications

- $\frac{\partial \Pi(p)}{\partial p_i} > (<) 0$ iff *i* is an output (input);
- y(·) and x(·) are homogeneous of degree 0;
- $\frac{\partial y_i}{\partial p_i} > 0$ and $\frac{\partial y_i}{\partial p_j} = \frac{\partial y_j}{\partial p_i}$, for all i,j=1,…,n.

2.4. Cost minimization

2.4.1. The cost minimization problem

- $c(y,w) = \underset{x}{Min}\, wx \ s.t.\ f(x) = y,\ x_i \geq 0,\ i = 1,...n-1$. The FOC's imply

$$w_i \geq \lambda \frac{\partial f(x^*)}{\partial x_i}, x_i^* \geq 0, (w_i - \lambda \frac{\partial f(x^*)}{\partial x_i}) x^*_i = 0, i = 1,...,n-1 \text{ and } f(x^*) = y.$$

- The solutions of the above problem are the *conditional input demands*, denoted by $x_i(\boldsymbol{w}, y)$.

- $\lambda^* = w_i \Big/ \frac{\partial f}{\partial x_i}(x^*)$ tells us how much the cost increases if we tighten the

constraint by requiring an extra unit of y, i.e., λ equals the marginal cost.

2.4.2. Implications of cost minimization: comparative statics

Given a list of input price vectors $\mathbf{w}^t$ and the associated optimal factor levels $\mathbf{x}^t$, t=1,…,T, an obvious necessary condition for cost minimization is that $\mathbf{w}^t\,\mathbf{x}^t \leq \mathbf{w}^t\,\mathbf{x}^s$ for all t,s such that $\mathbf{y}^s \geq \mathbf{y}^t$. This is the Weak Axiom of Cost Minimization (WACM) and implies that $\triangle\mathbf{w}\triangle\mathbf{x} \leq 0$.

3. The cost function

3.1. Definitions

- Short and long-term total costs, average costs (SAC and LAC), and marginal cost functions (SMC and LMC).

- Facts:
  1. AC are U-shaped;
  2. MC=AC at minAC;
  3. MC(1)=AC(1);
  4. if the technology exhibits constant returns to scale then AC=MC;
  5. SAC≥LAV (in the short run all factors cannot be adjusted in response to a change in input prices; thus, short run total cost must exceed long run total cost for any output level and the same is also true for average costs);
  6. SMC> or <LMC (they are equal when the output level is such that the fixed factors are optimally utilized; for lower levels of output there is too much of the fixed factor in the short run which makes the total cost higher than its long run value but the high level of the fixed factor makes it cheaper to produce an additional unit of output -SMC- than it would be once the fixed factor is adjusted to its long run level);
  7. SAC is tangent to LAC at the optimum (where SMC = LMC).

3.2. Properties

a. c(·) is increasing in y and nondecreasing in **w**;

b. c(·) is homogeneous of degree 1 in ***w***;

c. c(·) is concave in ***w***; intuition: by responding to the price change and adjusting the input mix, the cost is at least as low as if the firm did not respond;

d. c(·) is continuous in ***w***, for **w**$\gg$0;

e. *Shepard's lemma*: $\frac{\partial c(w,y)}{\partial w_i} = x_i(w,y),$ for all i=1,…,n, assuming that the derivative exists and $w_i > 0$.

Proofs: V, p. 72, 73, and 74

3.3. Implications

a. *a*. above implies: $\frac{\partial c(w,y)}{\partial w_i} = x_i(w,y) \geq 0$;

b. *b*. above implies that the $x_i(\cdot)$ are homogeneous of degree 0;

c. *c*. above implies that $\frac{\partial x_i}{\partial w_i} \leq 0$ and $\frac{\partial x_i}{\partial w_j} = \frac{\partial x_j}{\partial w_i}$, for all i,j=1,…,n.

4. Duality

4.1. Mathematical introduction

- A *half-space* is a set of the form $\{\mathbf{x}\in\mathbb{R}^n:\mathbf{p.x}\geq c\}$ for some $\mathbf{p}\in\mathbb{R}^n$, $\mathbf{p}\neq\mathbf{0}$, called the *normal vector* to the half-space;

- Its boundary $\{\mathbf{x}\in\mathbb{R}^n:\mathbf{p.x}=c\}$ is called a *hyperplane*;

- Half-spaces and hyperplanes are convex sets;

- Given a closed and convex set $K\subset\mathbb{R}^n$ and a point **x** outside K, the *separating hyperplane theorem* states that there is a half-space containing K and excluding **x**, i.e., there is a $\mathbf{p}\in\mathbb{R}^n$ and a $c\in\mathbb{R}$ such that $\mathbf{p.x}<c\leq\mathbf{p.x'}$, for all $\mathbf{x'}\in K$;

- **Idea**: a convex set may be equivalently described as the intersection of half-spaces that contain it; more generally, if the set K is not convex, the intersection of half-spaces is the smallest closed, convex set that contains K (known as the *closed convex hull* of K).

4.2. Duality in Production

The input requirement set V(y) describes the technology. Depending on the relative prices of inputs firms choose different input bundles along the isoquant.

Suppose that we do not know the underlying technology but can observe the input choices of a cost minimizing firm for all possible prices.

Suppose the bundle **x** is chosen at prices **w**. Then the set of all input bundles more expensive than this must include the true input requirement set. Let V* (y) be the set of input bundles that are at least as expensive as the bundle chosen at **w**, for all prices **w**. (V* (y) is the convex hull of V(y)).

If the technology is convex and monotonic it turns out that V* (y) = V(y).

In this case we can go from the cost function to the true technology. But even if the true technology is non-convex its cost function will coincide with that of V*(y), i.e. c*(**w**,y) = c(**w**,y).

$V(y) \subset V^*(y) \Rightarrow c^*(\mathbf{w},y) \leq c(\mathbf{w},y)$. Can V* (y) contain bundles, not in V(y), that are strictly cheaper? No. By definition V* (y) contains bundles that are at least as expensive as the optimal one in V(y) for any given **w**.

Thus the cost function summarizes the economically relevant information about the technology.

Furthermore, it turns out that:

- A differentiable function satisfying the properties for cost functions above is indeed a cost function for some technology.

- Functions satisfying the properties of the conditional demand functions - (i) homogeneity of degree 0 in prices and (ii) that the matrix of partial derivatives with respect to prices is symmetric and negative semidefinite - can be show to be a conditional demand function for some technology.

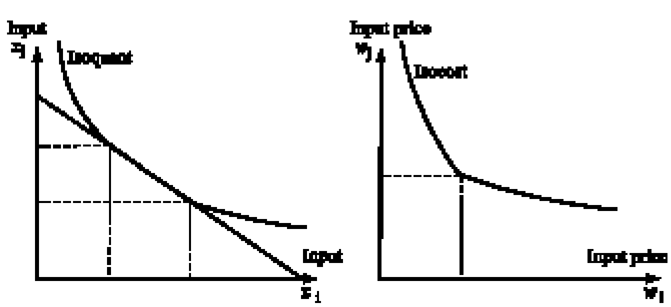
Going from costs to technology: See example in Varian, p 87.

Geometry of duality

The isocost curve is the set of input prices that allows us to produce a given output at the same cost. The slope of the isocost curve is simply, i.e.it equals the *ratio of the factor demands*. The slope of the isoquants is given by -TRS which must equal the *ratio of the input prices* in optimum.

A very curved isoquant means that large changes in factor prices lead to small changes in input choices. Thus the ratio of factor demands will remain relatively unchanged which means that the isocost curve is quite flat. Conversely, if the technology is linear (linear isoquant) we will only use the best input. Small changes in the prices of other inputs obviously has no effect on cost but a sufficiently large change will lead the firm to switch input, in which case only the price of the new input matters.

Consequently, the curvature of the isocost and the isoquant are inversely related.



In this section, we have shown that:

1. We can recover information on a firm's technology – as described by the input requirement set V(y) – using the intersection of half-spaces built by means of the cost function c(·), so that the cost function summarizes the economically relevant information about the technology;
2. A differentiable function satisfying the properties for cost functions above is indeed a cost function for some technology;
3. Functions satisfying the properties of the conditional demand functions (homogeneity of degree 0 in prices and symmetric and negative semi-definite matrix of partial derivatives with respect to prices) can be show to be a conditional demand function for some technology.

Proofs: V, p. 83-86

Example: V, p. 87