Lecture 5

Varian, Ch. 8; MWG, Chs. 3.E, 3.G, and 3.H

1 Summary of Lectures 1, 2, and 3: Production theory and duality

2 Summary of Lecture 4: Consumption theory

2.1 Preference orders

2.2 The utility function

2.3 The utility maximization problem

2.3.1 Solving the UMP

2.3.2 Walrasian demand

2.3.3 Indirect utility function

2.4 The expenditure minimization problem

2.4.1 Compensated or Hicksian demand

Instead of maximizing utility given a budget constraint we can consider the dual problem of minimizing the expenditure necessary to obtain a given utility level. Specifically, if we would like to reach the utility level that results in the first problem it turns out that the bundle that minimizes the cost of doing so coincides with the solution to the first problem.

The FOC for expenditure minimization imply the same relation between the prices and the marginal utilities as the FOC for utility maximization.

The solution to this problem is the optimal consumpion bundles as functions of p and u. Income is adjusted so the consumer can afford the cheapest possible bundle that yields u. These demand functions (one for each good) are called *compensated or Hicksian demand* functions and are denoted $h(p, u)$.

2.4.2 Expenditure function

The minimal expenditure necessary to reach u is the *expenditure function*:

$$\sum_i p_i \cdot h_i(p, u) = e(p, u)$$

Remark 1 *Local non-satiation*

> This assumption implies that $v(p, m)$ is strictly increasing in m. Thus we can derive the minimal expenditure necessary to reach u, $e(p, u)$, simply by inverting $v(p, m)$. It follows that $e(p, u)$ is strictly increasing in u.

Properties of the expenditure function

1. $e(p,u)$ is nondecreasing in p.
2. $e(p,u)$ is homogeneous of degree 1 in p.
3. $e(p,u)$ is concave in p.
4. $e(p,u)$ is continuous in p.
5. $\partial e(p,u)/\partial p_i = h_i(p,u)$.

Remark 2 *These are the same properties that cost functions have!*

2.4.3 Hicksian demand

Proposition 3 *Let $u()$ be a continuous utility function representing a locally non-satiated $\succeq$ in $\Re^k_+$. Then, for $p \gg 0$, $h(p,u)$ has the following properties:*

1. Homogeneous of degree 0 in p
2. No excess utility:$\forall x \in h(p,u),\ \ u(x) = u$.
3. Convexity/unicity

3 The expenditure minimization problem (cont.)

3.1 Important identities - Duality in consumption

Given the UMP:

$$\begin{aligned} v(p,m^*) &= \underset{x}{Max}\ u(x) \\ s.t.\ p\cdot x &\leq m^*, \end{aligned}$$

let x^* be the solution to this problem and let $u^* = u(x^*)$. Consider the EMP:

$$\begin{aligned} e(p,u^*) &= \underset{x}{Min}\ p\cdot x \\ s.t.\ u(x) &\geq u^*. \end{aligned}$$

In general, x^* is the solution to the EMP. This leads to:

1. $e(p, v(p,m)) \equiv m$.
2. $v(p, e(p,u)) \equiv u$.
3. $x_i(p,m) \equiv h_i(p, v(p,m))$.
4. $h_i(p,u) \equiv x_i(p, e(p,u))$.

Roy's identity Differentiating 2., we obtain Roy's identity:

$$x_i(p,m) = \frac{\frac{\partial v(p,m)}{\partial p_i}}{\frac{\partial v(p,m)}{\partial m}}, \text{ for } i = 1,...,k,\ p_i > 0 \text{ and } m > 0.$$

3.2 Money metric utility functions

As was noted above, local non satiation implies that $e(p,u)$ is strictly increasing in u. Since utility functions are only unique up to positive monotone transformations we can use the expenditure function to define $m(p,x) = e(p,u(x))$. For given p, this is a *money metric* utility function and, for given x, it is an expenditure function.

Similarly, we can define $\mu(p;q,m) = e(p,v(q,m))$ which measures the income required at prices p to be as well off as with the income m at prices q. This is a *money metric* indirect utility function; it is useful in welfare analysis.

4 Choice

4.1 Comparative statics of consumer behavior

The solution to the consumer's optimization problem gives us the optimal demand for goods as functions of prices and income, $x(p,m)$.

An *income expansion path* depicts how consumption changes with income and slopes upwards for normal goods. (Necessities & Luxury goods)

Price offer curves trace out how consumption changes as prices change. Demand decreases in price for ordinary good and increases for a Giffen good.

4.2 Income and substitution effects

The own substitution effect: The change in consumption caused by the change in relative prices keeping utility constant (by adjusting income).

The income effect: The difference in consumption between the above point and the new optimal consumption bundle.

- A normal good cannot be a Giffen good.
- The own substitution effect is always opposite to the price change.

4.2.1 The Slutsky equation

The Slutsky equation decomposes the demand change induced by a price change into two effects - the substitution and the income effect:

$$\frac{\partial x_j(p,m)}{\partial p_i} = \frac{\partial h_j(p,v(p,m))}{\partial p_i} - \frac{\partial x_j(p,m)}{\partial m} x_i(p,m).$$

4.2.2 Properties of demand functions

- Since the $e(p,m)$ is concave, the matrix of substitution terms is negative semi-definite.
  - Thus the diagonal terms - the own price effects - are negative.
  - The matrix of substitution terms is symmetric.

Remark 4 *Integrability: if a set of demand functions give rise to symmetric and negative semi-definite matrix of substitution terms then we can solve for the indirect utility function and the expenditure function. (c.f. the condition determining whether we can go from conditional demand functions to the technology).*

4.3 Revealed preference

Observe: (p^t, x^t) for some t. Suppose $p^t x^t \geq p^t x$, then $u(x^t) \geq u(x)$ and so $x^t R^D x$.

We say: x^t is directly revealed preferred to x.

We say: x^n is revealed preferred to x (denoted $x^t R x$) if there exists: $x^n R^D x^{n-1}, x^{n-1} R^D x^{n-2}, ..., x^1 R^D x$.

Weak Axiom of Revealed Preference: If $x^t R^D x^s$ and x^t is not equal to x^s, then it is not the case that $x^s R^D x^t$.

Strong Axiom of Revealed Preference: If $x^t R x^s$ and x^t is not equal to x^s, then it is not the case that $x^s R x^t$.

5 Demand

5.1 Homothetic utility

A homothetic utility function can be represented by a function that is homogenous of degree 1 (a monotonic transformation). A proportional increase in consumption of all goods then yields a proportional increase in utility. For given prices the same consumption mix is optimal regardless of income. Hence, the expenditure function can be expressed as $e(p,u) = e(p)u$ implying that $v(p,m) = v(p)m$ and $x_i(p,m) = x_i(p)m$.

5.2 Aggregation across consumers

Aggregate demand is a function of price and aggregate income if agents have *Gorman-type utility functions*: $v_i(p, m_i) = a_i(p) + b(p)m_i$. The crucial feature is that changes in income affects all consumers' behavior the same way. Therefore demand only depends on the aggregate income and not on how it is distributed among individuals. Homothetic and quasilinear utility functions have this property.

5.3 Convex preferences ensures continuity...

6 Consumers' surplus

6.1 Measuring welfare effects

6.1.1 The compensating variation (CV)

In general a policy change may affect both income and prices. Given that a change takes place what income compensation is required to leave the consumer as well of as before the change.

$$CV = m^1 - e(p^1, u^0) = \mu(p^1; p^1, m^1) - \mu(p^1; p^1, m^0)$$

where $\mu(q; p, m) = e(q, v(p, m))$. Suppose only one price changes and income remains constant, $m^0 = m^1$. Specifically, let p^1 fall from p_1^0 to p_1^1. In this case,

$$e(p^0, u^0) - e(p^1, u^0) = \int_{p_1^1}^{p_1^0} \frac{\partial e}{\partial p_1} dp_1 = \int_{p_1^1}^{p_1^0} h_1(p, u^0) dp_1.$$

6.1.2 The equivalent variation (EV)

Suppose prices and income remain the same. What income change would be necessary to give the consumer the same utility as he would have obtained if the price change from p_1^0 to p_1^1 had taken place?

By the same argument as above we can obtain:

$$EV = e(p^0, u^1) - e(p^1, u^1) = \int_{p_1^1}^{p_1^0} h_1(p, u^1) dp_1.$$

Note that the consumer surplus, CS, obeys EV > CS > CV.

6.1.3 Quasi-linear utility and no income effects

No income effects means that the consumption of the good depends only on the relative prices and not on income (provided that the income suffices to finance the desired quantity). Consequently the Hicksian demand curves and the Marshallian demand curve coincide and CV must equal EV.