Exercises from *Arbitrage Theory in Continuous Time (3:rd ed)* Chapters 4-5

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1 Exercises

Exercise 1.1 *Compute the stochastic differential dZ when*

(a) $Z(t) = e^{\alpha t}$,

(b) $Z(t) = \int_0^t g(s)dW(s)$, *where g is an adapted stochastic process.*

(c) $Z(t) = e^{\alpha W(t)}$

(d) $Z(t) = e^{\alpha X(t)}$, *where X has the stochastic differential*

$$dX(t) = \mu dt + \sigma dW(t)$$

(μ and σ are constants).

(e) $Z(t) = X^2(t)$, *where X has the stochastic differential*

$$dX(t) = \alpha X(t)dt + \sigma X(t)dW(t).$$

Exercise 1.2 *Compute the stochastic differential for Z when $Z(t) = \frac{1}{X(t)}$ and X has the stochastic differential*

$$dX(t) = \alpha X(t)dt + \sigma X(t)dW(t).$$

By using the definition $Z = X^{-1}$ you can in fact express the right hand side of dZ entirely in terms of Z itself (rather than in terms of X). Thus Z satisfies a stochastic differential equation. Which one?

Exercise 1.3 *Let $\sigma(t)$ be a given deterministic function of time and define the process X by*

$$X(t) = \int_0^t \sigma(s)dW(s). \tag{1}$$

Show that the characteristic function of $X(t)$ (for a fixed t) is given by

$$E\left[e^{iuX(t)}\right] = \exp\left\{-\frac{u^2}{2}\int_0^t \sigma^2(s)ds\right\}, \quad u \in R, \tag{2}$$

thus showing that $X(t)$ is normally distibuted with zero mean and a variance given by

$$Var[X(t)] = \int_0^t \sigma^2(s)ds.$$

Exercise 1.4 *Suppose that X has the stochastic differential*

$$dX(t) = \alpha X(t)dt + \sigma(t)dW(t),$$

where α is a real number whereas $\sigma(t)$ is any stochastic process. Determine the function $m(t) = E[X(t)]$.

Exercise 1.5 *Suppose that the process X has a stochastic differential*

$$dX(t) = \mu(t)dt + \sigma(t)dW(t),$$

and that $\mu(t) \geq 0$ with probability one for all t. Show that this implies that X is a submartingale.

Exercise 1.6 *A function $h(x_1, \ldots, x_n)$ is said to be* **harmonic** *if it satisfies the condition*

$$\sum_{i=1}^n \frac{\partial^2 h}{\partial x_i^2} = 0.$$

It is **subharmonic** *if it satisfies the condition*

$$\sum_{i=1}^n \frac{\partial^2 h}{\partial x_i^2} \geq 0.$$

Let $W_1, \ldots, W_n$ be independent standard Wiener processes, and define the process X by $X(t) = h(W_1(t), \ldots, W_n(t))$. Show that X is a martingale (submartingale) if h is harmonic (subharmonic).

Exercise 1.7 *The object of this exercise is to give an argument for the formal identity*

$$dW_1 \cdot dW_2 = 0,$$

when W_1 and W_2 are independent Wiener processes. Let us therefore fix a time t, and divide the interval $[0, t]$ into equidistant points $0 = t_0 < t_1 < \cdots < t_n = t$, where $t_i = \frac{i}{n} \cdot t$. We use the notation

$$\Delta W_i(t_k) = W_i(t_k) - W_i(t_{k-1}), \quad i = 1, 2.$$

Now define Q_n by

$$Q_n = \sum_{k=1}^{n} \Delta W_1(t_k) \cdot \Delta W_2(t_k).$$

Show that $Q_n \to 0$ in L^2, i.e. show that

$$\begin{aligned} E[Q_n] &= 0, \\ Var[Q_n] &\to 0. \end{aligned}$$

Exercise 1.8 *Let X and Y be given as the solutions to the following system of stochastic differential equations.*

$$\begin{aligned} dX &= \alpha X dt - Y dW, \quad X(0) = x_0, \\ dY &= \alpha Y dt + X dW, \quad Y(0) = y_0. \end{aligned}$$

Note that the initial values x_0, y_0 are deterministic constants.

(a) *Prove that the process R defined by $R(t) = X^2(t) + Y^2(t)$ is deterministic.*

(b) *Compute $E[X(t)]$.*

Exercise 1.9 *For a $n \times n$ matrix A, the* **trace** *of A is defined as*

$$tr(A) = \sum_{i=1}^{n} A_{ii}.$$

(a) *If B is $n \times d$ and C is $d \times n$, then BC is $n \times n$. Show that*

$$tr(BC) = \sum_{ij} B_{ij} C_{ji}.$$

(b) *With assumptions as above, show that*

$$tr(BC) = tr(CB).$$

(c) *Show that the multi dimensional Itô formula can be written*

$$df = \left\{ \frac{\partial f}{\partial t} + \sum_{i=1}^{n} \mu_i \frac{\partial f}{\partial x_i} + \frac{1}{2} tr\left[\sigma^{\star} H \sigma\right] \right\} dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \sigma_i dW_i$$

where H denotes the Hessian matrix

$$H_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}.$$

Exercise 1.10 *Show that the scalar SDE*

$$\begin{aligned} dX_t &= \alpha X_t dt + \sigma dW_t, \\ X_0 &= x_0, \end{aligned}$$

has the solution

$$X(t) = e^{\alpha t} \cdot x_0 + \sigma \int_0^t e^{\alpha(t-s)} dW_s, \tag{3}$$

by differentiating X as defined by eqn (3) and showing that X so defined actually satisfies the SDE.

Hint: *Write eqn (3) as*

$$X_t = Y_t + Z_t \cdot R_t,$$

where

$$\begin{aligned} Y_t &= e^{\alpha t} \cdot x_0, \\ Z_t &= e^{\alpha t} \cdot \sigma, \\ R_t &= \int_0^t e^{-\alpha s} dW_s, \end{aligned}$$

and first compute the differentials dZ, dY and dR. Then use the multidimensional Itô formula on the function $f(y, z, r) = y + z \cdot r$.

Exercise 1.11 *Let A be an $n \times n$ matrix, and define the matrix exponential e^A by the series*

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

This series can be shown to converge uniformly.

(a) *Show, by taking derivatives under the summation sign, that*

$$\frac{de^{At}}{dt} = Ae^{At}.$$

(b) *Show that*

$$e^0 = I,$$

where 0 *denotes the zero matrix, and* I *denotes the identity matrix.*

(c) *Convince yourself that if* A *and* B *commute, i.e.* $AB = BA$*, then*

$$e^{A+B} = e^A \cdot e^B = e^B \cdot e^A.$$

Hint: *Write the series expansion in detail.*

(d) *Show that* e^A *is invertible for every* A*, and that in fact*

$$\left[e^A\right]^{-1} = e^{-A}.$$

(e) *Show that for any* A*,* t *and* s

$$e^{A(t+s)} = e^{At} \cdot e^{As}$$

(f) *Show that*

$$\left(e^A\right)^\star = e^{A^\star}$$

Exercise 1.12 *Consider the* n*-dimensional linear SDE*

$$\begin{cases} dX_t &= (AX_t + b_t)\, dt + \sigma_t dW_t, \\ X_0 &= x_0 \end{cases} \tag{4}$$

where A *is an* $n \times n$ *matrix,* b *is an* R^n*-valued deterministic function (in column vector form),* σ *is a deterministic* $n \times d$ *deterministic matrix valued function, and* W *an* d*-dimensional Wiener process. Show that the solution of this equation is given by*

$$X_t = e^{At}x_0 + \int_0^t e^{A(t-s)} b_s ds + \int_0^t e^{A(t-s)} \sigma_s dW_s. \tag{5}$$

Exercise 1.13 *Consider again the linear SDE (4). Show that the expected value function*

$$m(t) = E[X(t)]$$

, and the covariance matrix

$$C(t) = \{Cov(X_i(t), X_j(t)\}_{i,j}$$

are given by

$$\begin{aligned} m(t) &= e^{At}x_0 + \int_0^t e^{A(t-s)} b(s) ds, \\ C(t) &= \int_0^t e^{A(t-s)} \sigma(s) \sigma^\star(s) e^{A^\star(t-s)} ds, \end{aligned}$$

where * *denotes transpose.*

Hint: Use the explicit solution above, and the fact that

$$C(t) = E\left[X_t X_t^\star\right] - m(t)m^\star(t).$$

Geometric Brownian motion (GBM) constitutes a class of processes which is closed under a number of nice operations. Here are some examples.

Exercise 1.14 *Suppose that X satisfies the SDE*

$$dX_t = \alpha X_t dt + \sigma X_t dW_t.$$

Now define Y by $Y_t = X_t^\beta$, where β is a real number. Then Y is also a GBM process. Compute dY and find out which SDE Y satisfies.

Exercise 1.15 *Suppose that X satisfies the SDE*

$$dX_t = \alpha X_t dt + \sigma X_t dW_t,$$

and Y satisfies

$$dY_t = \gamma Y_t dt + \delta Y_t dV_t,$$

where V is a Wiener process which is independent of W. Define Z by $Z = \frac{X}{Y}$ and derive an SDE for Z by computing dZ and substituting Z for $\frac{X}{Y}$ in the right hand side of dZ. If X is nominal income and Y describes inflation then Z describes real income.

Exercise 1.16 *Suppose that X satisfies the SDE*

$$dX_t = \alpha X_t dt + \sigma X_t dW_t,$$

and Y satisfies

$$dY_t = \gamma Y_t dt + \delta Y_t dW_t.$$

Note that now both X and Y are driven by the same Wiener process W. Define Z by $Z = \frac{X}{Y}$ and derive an SDE for Z.

Exercise 1.17 *Suppose that X satisfies the SDE*

$$dX_t = \alpha X_t dt + \sigma X_t dW_t,$$

and Y satisfies

$$dY_t = \gamma Y_t dt + \delta Y_t dV_t,$$

where V is a Wiener process which is independent of W. Define Z by $Z = X \cdot Y$ and derive an SDE for Z. If X describes the price process of, for example, IBM in US\$ and Y is the currency rate SEK/US\$ then Z describes the dynamics of the IBM stock expressed in SEK.

Exercise 1.18 *Use a stochastic representation result in order to solve the following boundary value problem in the domain* $[0, T] \times R$.

$$\begin{aligned} \frac{\partial F}{\partial t} + \mu x \frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 F}{\partial x^2} &= 0, \\ F(T, x) &= \ln(x^2). \end{aligned}$$

Here μ *and* σ *are assumed to be known constants.*

Exercise 1.19 *Consider the following boundary value problem in the domain* $[0, T] \times R$.

$$\begin{aligned} \frac{\partial F}{\partial t} + \mu(t,x) \frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2(t,x) \frac{\partial^2 F}{\partial x^2} + k(t,x) &= 0, \\ F(T, x) &= \Phi(x). \end{aligned}$$

Here μ, σ, k *and* Φ *are assumed to be known functions.*

Prove that this problem has the stochastic representation formula

$$F(t,x) = E_{t,x}\left[\Phi(X_T)\right] + \int_t^T E_{t,x}\left[k(s, X_s)\right] ds,$$

where as usual X *has the dynamics*

$$\begin{aligned} dX_s &= \mu(s, X_s)ds + \sigma(s, X_s)dW_s, \\ X_t &= x. \end{aligned}$$

Hint: Define X as above, assume that F actually solves the PDE and consider the process $Z_s = F(s, X_s)$.

Exercise 1.20 *Use the result of the previous exercise in order to solve*

$$\begin{aligned} \frac{\partial F}{\partial t} + \frac{1}{2}x^2 \frac{\partial^2 F}{\partial x^2} + x &= 0, \\ F(T, x) &= \ln(x^2). \end{aligned}$$

Exercise 1.21 *Consider the following boundary value problem in the domain* $[0, T] \times R$.

$$\begin{aligned} \frac{\partial F}{\partial t} + \mu(t,x) \frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2(t,x) \frac{\partial^2 F}{\partial x^2} + r(t,x)F &= 0, \\ F(T, x) &= \Phi(x). \end{aligned}$$

Here $\mu(t,x)$, $\sigma(t,x)$, $r(t,x)$ *and* $\Phi(x)$ *are assumed to be known functions. Prove that this problem has a stochastic representation formula of the form*

$$F(t,x) = E_{t,x}\left[\Phi(X_T) e^{\int_t^T r(s, X_s) ds}\right],$$

by considering the process $Z_s = F(s, X_s) \times \exp\left[\int_t^s r(u, X_u)du\right]$ *on the time interval* $[t, T]$.

Exercise 1.22 *Solve the boundary value problem*

$$\begin{aligned} \frac{\partial F}{\partial t}(t,x,y) + \frac{1}{2}\sigma^2 \frac{\partial^2 F}{\partial x^2}(t,x,y) + \frac{1}{2}\delta^2 \frac{\partial^2 F}{\partial y^2}(t,x,y) &= 0, \\ F(T,x,y) &= xy. \end{aligned}$$

Exercise 1.23 *Go through the details in the derivation of the Kolmogorov forward equation.*

Exercise 1.24 *Consider the SDE*

$$dX_t = \alpha dt + \sigma dW_t,$$

where α and σ are constants.

(a) *Compute the transition density $p(s,y;t,x)$, by solving the SDE.*

(b) *Write down the Fokker–Planck equation for the transition density and check the equation is indeed satisfied by your answer in (a).*

Exercise 1.25 *Consider the standard GBM*

$$dX_t = \alpha X_t dt + \sigma X_t dW_t$$

and use the representation

$$X_t = X_s \exp\left\{ \left[\alpha - \frac{1}{2}\sigma^2\right](t-s) + \sigma\left[W_t - W_s\right] \right\}$$

in order to derive the transition density $p(s,y;t,x)$ of GBM. Check that this density satisfies the Fokker-Planck equation.