II. B. O MODELO DE UM ÍNDICE E O MODELO DE ÍNDICES MÚLTIPLOS [*FACTOR MODELS*]

Bibliografia:

Bodie, Kane e Marcus, capítulo 10

Elton, Gruber, Bown, e Goetzmann, capítulos 7 e 8.

Afonso, Barros, Calado, Borges, Garcia e Relvas, capítulos 3 e 4.

Pires, capítulo 6.

- Information about assets:
  - Expected values
  - Variance-covariance matrix

⇨ **MARKOWITZ FRONTIER**

- Assuming investors are risk averse and use mean-variance criterion to decide investment
  - Estimation aversion coefficient (allows to characterize indifference curves)

OPTIMUM PORTFOLIO

However, a lot of parameters are needed…

- N expected returns;
- N^2 elements of the variance-covariance matrix
  - N(N-1)/2 covariances (or correlation coefficients);
  - N variances.

Total: 2N+N(N-1)/2 estimates

For instance, an analyst that follows between 150 and 250 assets needs to estimate between 11 475 e 31 625 parameters.

FACTOR MODELS

New modelization to simplify the variance-covariance matrix

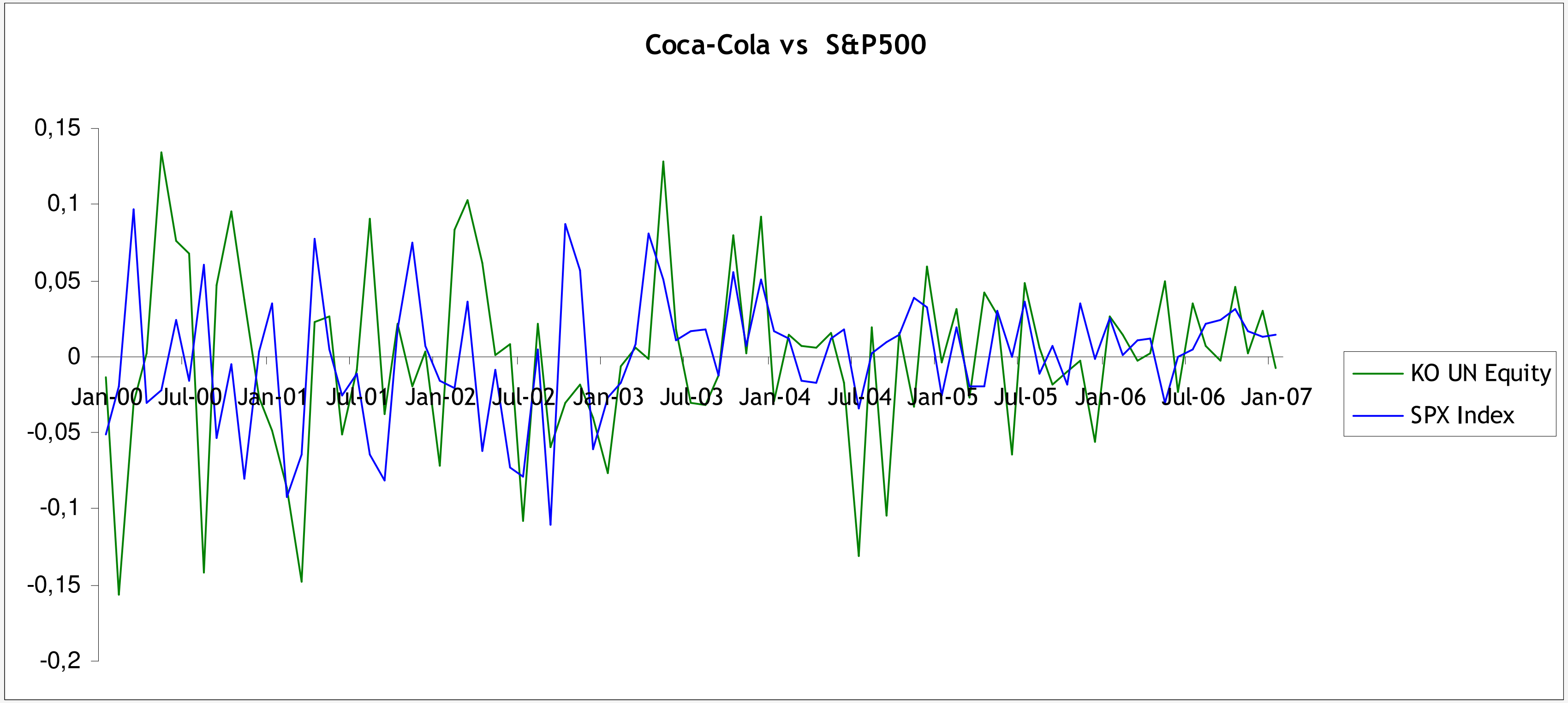
1. SINGLE-FACTOR MODELS

A single factor explains all co-movements in returns

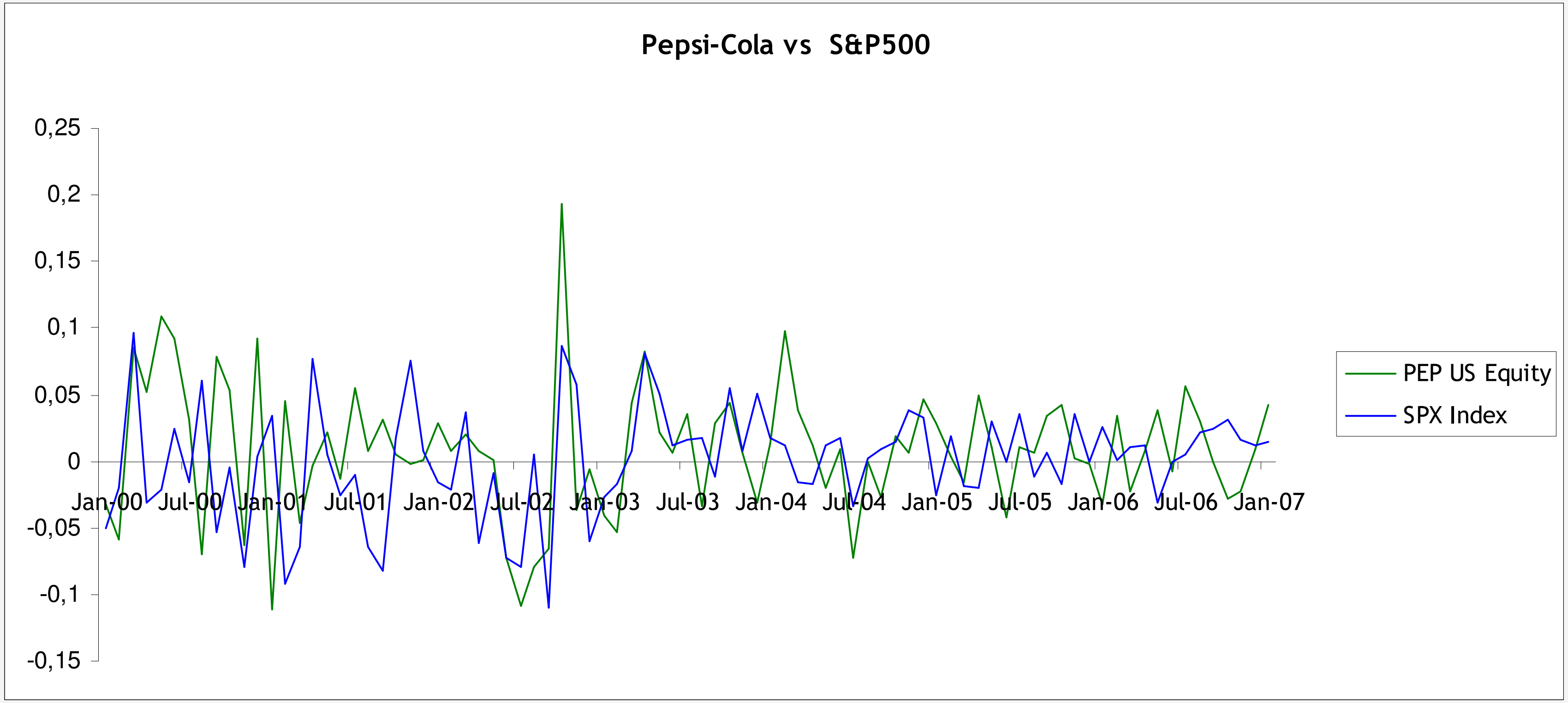
2. MULTIFACTOR MODELS

Several factors explains all co-movements in returns

SINGLE-FACTOR MODEL



SINGLE-FACTOR MODEL



SINGLE-FACTOR MODEL

Underlying Idea to the Single-Factor Model

Each security responds, in some cases more and in other cases less, to changes of a single factor, which is usually taken to be the market portfolio. Hence, it is assumed that the variability in the factor accounts for all the co-movement that is observed among assets.

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The covariance matrix reflects the fact that all stocks respond to this single factor

SINGLE-FACTOR MODEL

f : (macroeconomic) factor with return R_{ft}

$$R_{it} = \alpha_i + \beta_i R_{ft} + \epsilon_{it}$$

where,

α_i : Constant → component of the return of asset *i* that is independent of the return of the factor

R_{ft} : Random variable → return of factor *f*

β_i : Constant → sensitivity of the return of asset *i* to the return of the factor

ϵ_{it} : Random variable

SINGLE-FACTOR MODEL

$$R_{it} = \alpha_i + \beta_i R_{ft} + \epsilon_{it}$$

Assumptions:

- $E[\epsilon_i] = 0$ $\quad \text{var}(\varepsilon_i) = \sigma^2_{\varepsilon_i}$
- $cov(\epsilon_i, \epsilon_j) = E[\epsilon_i \epsilon_j] = 0$ ← MAIN ASSUMPTION.
- $cov(\epsilon_i, R_f) = E[\epsilon_i (R_f - E[R_f])] = 0$

SINGLE-FACTOR MODEL

EXPECTED RETURN OF ASSET *i*

$$E[R_i] = \alpha_i + \beta_i E[R_f]$$

VARIANCE OF ASSET *i*

$$\sigma_i^2 = \beta_i^2 \sigma_f^2 + \sigma_{\epsilon_i}^2$$

Risk of asset *i* can be decomposed in **two components**:

- $\beta_i^2 \sigma_f^2$ Systematic risk.
- $\sigma_{\epsilon_i}^2$ Specific risk

COVARIANCE BETWEEN ASSET *i* and *j*

$$\sigma_{ij} = \beta_i \beta_j \sigma_f^2$$

SINGLE-FACTOR MODEL

VARIANCE-COVARIANCE MATRIX FOR N ASSETS

$$\Sigma = \begin{bmatrix} \beta_1^2\sigma_f^2+\sigma_{\epsilon_1}^2 & \beta_1\beta_2\sigma_f^2 & \cdots & \beta_1\beta_N\sigma_f^2 \\ \beta_2\beta_1\sigma_f^2 & \beta_2^2\sigma_f^2+\sigma_{\epsilon_2}^2 & & \beta_2\beta_N\sigma_f^2 \\ \vdots & \vdots & & \vdots \\ \beta_N\beta_1\sigma_f^2 & \beta_N\beta_2\sigma_f^2 & \cdots & \beta_N^2\sigma_f^2+\sigma_{\epsilon_N}^2 \end{bmatrix}$$

$$= \begin{bmatrix} \beta_1^2 & \beta_1\beta_2 & \cdots & \beta_1\beta_N \\ \beta_2\beta_1 & \beta_2^2 & & \beta_2\beta_N \\ \vdots & \vdots & & \vdots \\ \beta_N\beta_1 & \beta_N\beta_2 & \cdots & \beta_N^2 \end{bmatrix}\sigma_f^2 + \begin{bmatrix} \sigma_{\epsilon_1}^2 & 0 & \cdots & 0 \\ 0 & \sigma_{\epsilon_2}^2 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \sigma_{\epsilon_N}^2 \end{bmatrix}$$

SINGLE-FACTOR MODEL

Not so many parameters are requires anymore…

- N estimates for α_i ;
- N estimates for β_i ;
- N estimates for $\sigma^2_{\epsilon_i}$;
- One estimate for $E\left[R_f\right]$ and another for σ^2_f .

Total: 3N+2 estimates

An analyst who follows between 150 and 250 assets, should estimate between 452 and 752 parameters and not 11475-31625, as stressed initially.

SINGLE-FACTOR MODEL

PORTFOLIO

$$x_i \quad \text{with} \quad \sum_{i=1}^{N} x_i = 1$$

Define

$$\alpha_P = \sum_{i=1}^{N} x_i \alpha_i$$

$$\beta_P = \sum_{i=1}^{N} x_i \beta_i$$

SINGLE-FACTOR MODEL

EXPECTED RETURN OF THE PORTFOLIO

$$E\left[R_P\right] = \sum_{i=1}^{N} x_i E\left[R_i\right]$$

As, $E\left[R_i\right] = \alpha_i + \beta_i E\left[R_f\right]$

$$E[R_P] = \sum_{i=1}^{N} x_i \alpha_i + \sum_{i=1}^{N} x_i \beta_i E[R_F]$$

$$\boxed{E\left[R_P\right] = \alpha_P + \beta_P E\left[R_f\right]}$$

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SINGLE-FACTOR MODEL

VARIANCE OF THE PORTFOLIO

$$\sigma_P^2 = \sum_{i=1}^{N} x_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} x_i x_j \sigma_{ij}$$

As, $\sigma_i^2 = \beta_i^2 \sigma_f^2 + \sigma_{\epsilon_i}^2$ and $\sigma_{ij} = \beta_i \beta_j \sigma_f^2$

$$\sigma_P^2 = \sum_{i=1}^{N} x_i^2 \beta_i^2 \sigma_f^2 + \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} x_i x_j \beta_i \beta_j \sigma_f^2 + \sum_{i=1}^{N} x_i^2 \sigma_{\epsilon_i}^2$$

$$\sigma_P^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} x_i x_j \beta_i \beta_j \sigma_f^2 + \sum_{i=1}^{N} x_i^2 \sigma_{\epsilon_i}^2$$

$$\sigma_P^2 = \sigma_f^2 \sum_{i=1}^{N} x_i \beta_i \sum_{j=1}^{N} x_j \beta_j + \sum_{i=1}^{N} x_i^2 \sigma_{\varepsilon_i}^2$$

$$\boxed{\sigma_P^2 = \beta_P^2 \sigma_f^2 + \sum_{i=1}^{N} x_i^2 \sigma_{\epsilon_i}^2}$$

SINGLE-FACTOR MODEL and THE DIVERSIFICATION EFFECT

When the number of assets (N) increases the importance of the residual risk (variance) decreases.

Number of Securities	Residual Risk (Variance) Expressed as a Percent of the Residual Risk Present in a One-Stock Portfolio with σ^2_{ei} a Constant
1	100
2	50
3	33
4	25
5	20
10	10
20	5
100	1
1000	0.1

SINGLE-FACTOR MODEL and THE DIVERSIFICATION EFFECT

- Consider that the amount invested in each asset is the same
- Then, $x_i = 1/N$ and the variance of the portfolio becomes

$$\sigma_P^2 = \beta_P^2 \sigma_f^2 + \underbrace{\frac{1}{N} \sum_{i=1}^{N} \frac{\sigma_{\epsilon_i}^2}{N}}_{\text{Mean residual risk}}$$

- When $N \to \infty$ the term of the mean residual risk tends to zero and the variance of the portfolio approaches

$$\sigma_P^2 \to \beta_P^2 \sigma_f^2$$

SINGLE-FACTOR MODEL and THE DIVERSIFICATION EFFECT

In a well diversified economy the specific risk is eliminated and the risk of the portfolio approaches

$$\beta_P \sigma_f$$

SINGLE-FACTOR MODEL and THE DIVERSIFICATION EFFECT

IMPLICATIONS FOR INDIVIDUAL ASSETS

$$\sigma_i^2 = \beta_i^2 \sigma_f^2 + \sigma_{\epsilon_i}^2$$

As the specific (or residual risk) $\sigma_{\epsilon_i}^2$ can be eliminated by diversification,

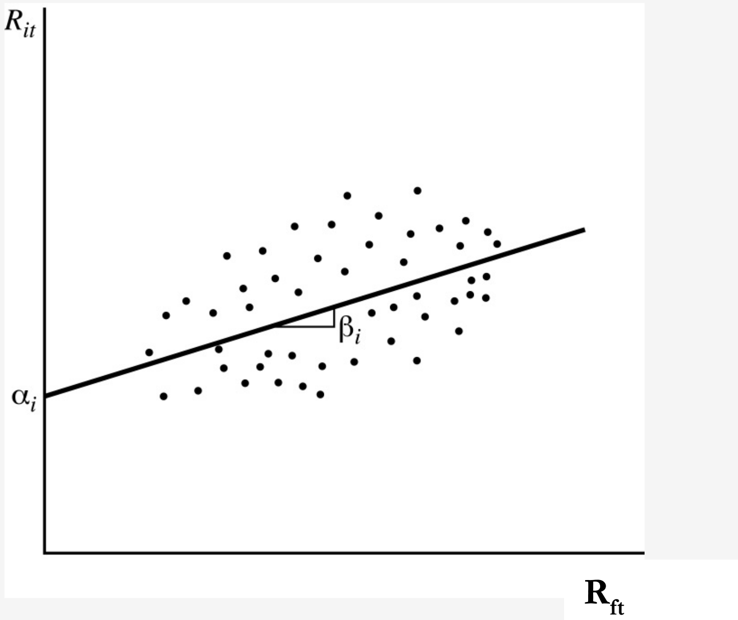
β_i is commonly used as a measure of risk of asset i

SINGLE-FACTOR MODEL

:: ESTIMATION OF THE BETAS

Linear regression - Minimize the sum of squared residuals

$$\min_{\hat{\alpha}_i,\hat{\beta}_i} \sum_{t=1}^{T} [R_{it} - (\hat{\alpha}_i + \hat{\beta}_i R_{ft})]^2$$



1st order conditions:

$$\hat{\beta}_i = \frac{\text{cov}(R_i, R_f)}{\text{var}(R_f)} = \frac{\sigma_{if}}{\sigma_f^2} = \rho_{if}\frac{\sigma_i}{\sigma_f} =$$

$$= \frac{\sum_{t=1}^{T}[(R_{it} - \overline{R}_i)(R_{ft} - \overline{R}_f)]}{\sum_{t=1}^{T}(R_{ft} - \overline{R}_f)^2}$$

$$\hat{\alpha}_i = \overline{R}_i - \hat{\beta}_i \overline{R}_f$$

$$E[R_i] = \alpha_i + \beta_i E[R_f]$$

SINGLE-FACTOR MODEL

:: ESTIMATION OF THE BETAS

Example Beta Estimation using *excel*

Tools / Data Analysis

SUMMARY OUTPUT

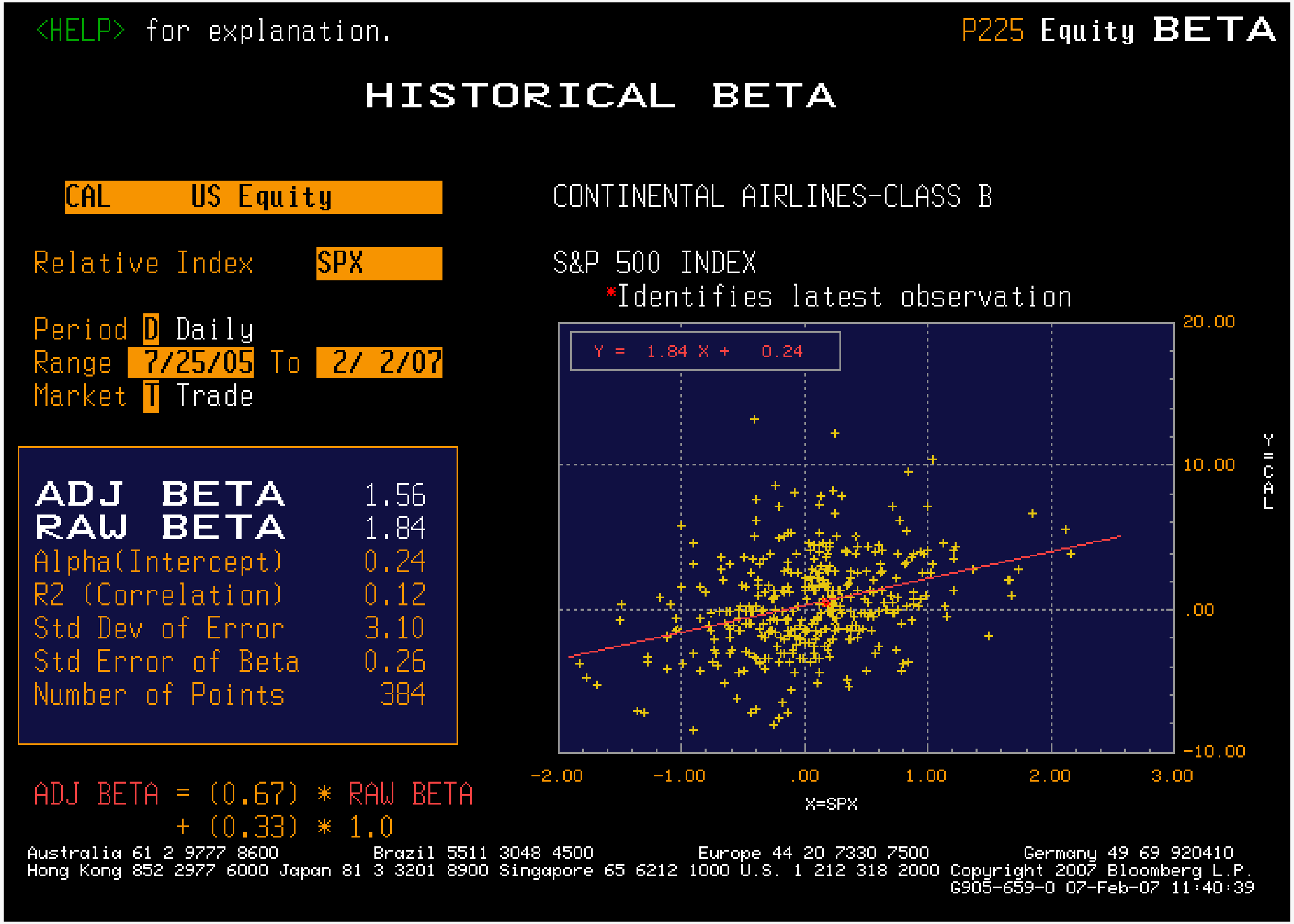
Regression Statistics	
Multiple R	0,260942881
R Square	0,068091187
Adjusted R Square	0,05686337
Standard Error	0,04799972
Observations	85

R^2= fitting quality measure

In simple Linear regression:

$$R^2 = \rho_{if}^{\ 2} = \left(\hat{\beta}_i \frac{\sigma_f}{\sigma_i} \right)^2$$

	Coefficients	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*	*Lower 95,0%*	*Upper 95,0%*
Intercept	0,008272732	0,005206842	1,588819658	0,115903	-0,002083466	0,01862893	-0,002083466	0,01862893
X Variable 1	0,314462642	0,127694229	2,462622201	0,015861	0,060483962	0,568441322	0,060483962	0,568441322


<HELP> for explanation.
P225 Equity BETA
HISTORICAL BETA
CAL US Equity
CONTINENTAL AIRLINES-CLASS B
Relative Index SPX
S&P 500 INDEX
*Identifies latest observation
Period D Daily
Range 7/25/05 To 2/ 2/07
Market T Trade
Y = 1.84 X + 0.24
ADJ BETA 1.56
RAW BETA 1.84
Alpha(Intercept) 0.24
R2 (Correlation) 0.12
Std Dev of Error 3.10
Std Error of Beta 0.26
Number of Points 384
20.00
10.00
.00
-10.00
Y=CAL
-2.00
-1.00
.00
1.00
2.00
3.00
X=SPX
ADJ BETA = (0.67) * RAW BETA
+ (0.33) * 1.0
Australia 61 2 9777 8600 Brazil 5511 3048 4500 Europe 44 20 7330 7500 Germany 49 69 920410
Hong Kong 852 2977 6000 Japan 81 3 3201 8900 Singapore 65 6212 1000 U.S. 1 212 318 2000 Copyright 2007 Bloomberg L.P.
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MULTIFACTOR MODELS

THE ONE FACTOR MODEL:

The common movement of the returns of the different assets is driven by a unique factor

Are there other factors that may have na impact on the common movement of the returns?

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MULTI FACTOR MODELS

But…

- ✓ Economic factors? Per industry?
- ✓ Not so difficult to find factors that explain the time evolution of prices. And the variance-covariance?… Excess of parameters to be estimated, again?

MULTIFACTOR MODELS

K factors

$$R_{it} = \alpha_i + \beta_{if_1} R_{f_1 t} + \ldots + \beta_{if_K} R_{f_K t} + \epsilon_{it}$$

where,

α_i : Constant → component of the return of asset *i* that is independent of the return of the factors

$R_{f_k t}$: Random variable → return of factor *k=1,…,K*

β_{if_k} : Constant → sensitivity of the return of asset *i* to the return of the factor f_k

ϵ_{it} : Random variable

MULTIFACTOR MODELS

$$R_{it} = \alpha_i + \beta_{if_1} R_{f_1 t} + ... + \beta_{if_K} R_{f_K t} + \epsilon_{it}$$

Assumptions:

- $E[\epsilon_i] = 0 \qquad \mathrm{var}(\varepsilon_i) = \sigma^2_{\varepsilon_i}$
- $cov(\epsilon_i, \epsilon_j) = E[\epsilon_i \epsilon_j] = 0, \quad i \neq j$
- $cov(\epsilon_i, R_{f_k}) = E[\epsilon_i (R_{f_k} - E[R_{f_k}])] = 0, \quad \forall k, i$
- $E[(R_{f_k} - E[R_{f_k}])(R_{f_h} - E[R_{f_h}])] = 0, \quad k \neq h$

Ortogonal Factors

MULTIFACTOR MODELS

EXPECTED RETURN OF ASSET *i*

$$E\left[R_i\right] = \alpha_i + \sum_{i=1}^{K} \beta_{if_k} E\left[R_{f_k}\right]$$

VARIANCE OF ASSET *i*

$$\sigma_i^2 = \sum_{k=1}^{K} \beta_{if_k}^2 \sigma_{f_k}^2 + \sigma_{\epsilon_i}^2$$

COVARIANCE BETWEEN ASSET *i* and *j*

$$\sigma_{ij} = \sum_{k=1}^{K} \beta_{if_k} \beta_{jf_k} \sigma_{f_k}^2$$

MULTIFACTOR MODELS

Which Factors? E.g.:

- GDP Growth
- Oil price;
- Inflation rate;
- *Spread* between short term and long term rates;
- Return on goverment debt (*títulos da dívida pública*);
- Market Factor
- Industry Factor
- Market Capitalization;
- …

MULTIFACTOR MODELS

Fama & French (83):

In their work they argue for a model of three factors

- Market Factor: *e.g.* S&P500 return
- Market Capitalization
- Value-Growth Factor:

firm's accounting value over firm's market value

http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html

http://www.barra.com/support/models/equity/equity_factor.asp

MULTIFACTOR MODELS

Required information… Estimation by multiple linear regression

- N estimates for α_i and N estimates for σ_{ε_i};
- NL estimates for β_{ij};
- L estimates for E[R] of each factor and L estimates for the variance of each factor.

Total: 2N+2L+NL estimates

For instance, an analyst that follows between **150 and 250** assets and consider 10 factors needs to estimate between **1 820 and 3 020** parameters. In the single factor would need between **452 and 752** parameters and in the Markowitz frontier between **11475 and 31625**.