Lecture 4

Varian, Ch. 7; MWG, Chs. 1.B, 3.B, 3.C, and 3.D

1 Summary of Lectures 1 and 2: Technology and profit maximization

2 Summary of Lecture 3: The cost function and duality

2.1 Definitions

Short and long-term total costs, average costs (SAC and LAC), and marginal cost functions (SMC and LMC).

2.2 Properties of the cost function

$c(\cdot)$ is increasing in y and nondecreasing in w; $c(\cdot)$ is homogeneous of degree 1 in w; $c(\cdot)$ is concave in w; continuous in w; Shepard's lemma.

2.3 Implications

2.4 Duality

2.4.1 Mathematical introduction

2.4.2 Duality in Production

- We can recover information on a firm's technology – as described by the input requirement set V(y) – using the intersection of half-spaces built by means of the cost function $c(\cdot)$, so that the cost function summarizes the economically relevant information about the technology;

- A differentiable function satisfying the properties for cost functions above is indeed a cost function for some technology;

- Functions satisfying the properties of the conditional demand functions (homogeneity of degree 0 in prices and symmetric and negative semi-definite matrix of partial derivatives with respect to prices) can be show to be a conditional demand function for some technology.

2.4.3 Geometry of duality

The isocost curve is the set of input prices that allows us to produce a given output at the same cost. The slope of the isocost curve is simply, i.e.it equals the ratio of the factor demands. The slope of the isoquants is given by -TRS which must equal the ratio of the input prices in optimum.

A very curved isoquant means that large changes in factor prices lead to small changes in input choices. Thus the ratio of factor demands will remain relatively unchanged which means that the isocost curve is quite flat. Conversely, if the technology is linear (linear isoquant) we will only use the best input. Small changes in the prices of other inputs obviously has no effect on cost but a sufficiently large change will lead the firm to switch input, in which case only the price of the new input matters.

Consequently, the curvature of the isocost and the isoquant are inversely related.

3 Consumption theory

3.1 Preference orders

In order to examine the optimal choice of the consumer, we need to be able to describe the consumer's preferences over different consumption bundles in a systematic way.

Definition 1 $X \subseteq \Re^k_+$ *is the set of consumption bundles.*

We assume X is closed and convex.

Definition 2 $x \succeq y$ *means that the bundle* x *is weakly preferred to* y*.*

Assumptions:

1. Completeness: for any pair of bundles x and y in X, either $x \succeq y$ or $y \succeq x$
2. Reflexivity: for any bundle x in X, $x \succeq x$
3. Transitivity: for any bundles x, y, z in X, if $x \succeq y$ and $y \succeq z$, then $x \succeq z$
4. Continuity: for any bundle x in X, the better set $\{y \in X : y \succeq x\}$ and the worse set $\{y \in X : y \succeq x\}$ are closed
5. Weak monotonicity: for any bundles x, y in X such that $x \geq y$, we have $x \succeq y$
6. Strong monotonicity: for any bundles x, y in X such that $x \geq y$ and $x \neq y$, we have $x \succeq y$
7. Convexity: given any bundle x, its better set (the set of bundles preferred or indifferent to x') is convex.
8. Strict convexity: given any bundle x, its better set (the set of bundles preferred or indifferent to x') is strictly convex.

Remark 3 *Preferences are rational iff they satisfy 1. and 3..*

3.2 The utility function

Definition 4 *A utility function is a function u assigning a real number to each consumption bundle x so that* $x \succeq y$ *iff* $u(x) \geq u(y)$

A utility function is an ordinal function and it is unique up to a positive monotonic transformation.

What is needed in order to ensure that consumer preferences can be represented by a utility function?

Theorem 5 *If* $\succeq$ *satisfy properties 1-4 e 6, then there exists a continuous utility function* $u : \Re^k_+ \rightarrow \Re$ *that represents* $\succeq$.

Proof. ... ■

Remark 6 *4. rules out lexicographic preferences.*

Remark 7 *6. and 8. imply that* $u(x)$ *is strictly quasi-concave.*

Definition 8 *Marginal Rate of Substitution (MRS): given* $u(x_1, x_2, ...x_k)$*, the MRS between good* j *and* i *is given by* $\frac{dx_i}{dx_j} = -\frac{\frac{\partial u}{\partial x_j}(x_1,x_2,...x_k)}{\frac{\partial u}{\partial x_i}(x_1,x_2,...x_k)}$.

Remark 9 *The MRS is independent of the utility function that is used to represent the preferences.*

Remark 10 *Assumptions on* $\succeq$ *reflect themselves on* $u(\cdot)$*:*

1. $\succeq$ monotonous implies $u(\cdot)$ increasing, i.e., $u(x) > u(y)$ if $x \gg y$
2. $\succeq$ convex implies $u(\cdot)$ quasi-concave
3. $\succeq$ strictly convex implies $u(\cdot)$ strictly quasi-concave
4. $\succeq$ homothetic iff $\exists u(\cdot)$ homogeneous of degree 1, i.e., $u(tx) = tu(x), \forall t > 0$, that represents it
5. $\succeq$ quasilinear iff $\exists u(\cdot)$ such that $u(x) = x_1 + \phi(x_2, x_3, ...x_k)$ that represents it

3.3 The utility maximization problem

Assumptions:
- $\succeq$ are rational, continuous, locally non-satiated
- $u(\cdot)$ is a continuous function that represents $\succeq$
- $X = \Re^k_+$

The consumer faces the following utility maximization problem (UMP):

$$\begin{aligned} v(p,m) &= \underset{x_1,x_2,...x_k}{Max}\ u(x_1,x_2,...x_k) \\ s.t. \sum_i p_i x_i &\leq m,\ x_i \geq 0,\ \forall i = 1,...,k. \end{aligned}$$

Definition 11 *Budget Set:* $B_{p,m} = \{x \in \Re^k_+ : p \cdot x \leq m\}$.

Theorem 12 *Let* $p \gg 0$ *and let* $u()$ *be a continuous utility function. Then, the utility maximization problem has a solution.*

From the UMP, we obtain:

1. The optimal consumption choices, or the demand, which depends on prices and income, i.e., $x^*(p,m)$(also called Walrasian or Marshallian demand)
2. The indirect utility function $v(p,m)$: by substituting the optimal consumption bundles $x^*(p,m)$ in $u(x)$;thus, the indirect utility function gives the maximum utility

that can be achieved for certain prices and a certain income.

3.4 Solving the UMP

In an interior optimum, the MRS between any two goods equals the ratio of their prices.

3.5 Walrasian demand

Proposition 13 *Let* $u()$ *be a continuous utility function representing a locally non-satiated* $\succeq$ *in* $\Re^k_+$*. Then,* $x^*(p,m)$ *has the following properties:*

1. Homogeneous of degree 1 in (p,m)
2. Walras' law: $p \cdot x = m, \forall x \in x^*(p,m)$
3. Convexity/unicity

3.6 Indirect utility function

Properties of the indirect utility function:

1. $v(p,m)$ is nonincreasing in p and nondecreasing in m.
2. $v(p,m)$ is homogeneous of degree 0 in (p,m).
3. $v(p,m)$ is quasiconvex in p.
4. $v(p,m)$ is continuous in $p >> 0$, $m > 0$.

Remark 14 *If* $\succeq$ *satisfies local non-satiation,* $v(p,m)$ *is strictly increasing in* m*.*

3.7 The expenditure minimization problem

Instead of maximizing utility given a budget constraint we can consider the dual problem of minimizing the expenditure necessary to obtain a given utility level. Specifically, if we would like to reach the utility level that results in the first problem it turns out that the bundle that minimizes the cost of doing so coincides with the solution to the first problem.

The FOC for expenditure minimization imply the same relation between the prices and the marginal utilities as the FOC for utility maximization.

The solution to this problem is the optimal consumpion bundles as functions of p and u. Income is adjusted so the consumer can afford the cheapest possible bundle that yields u. These demand functions (one for each good) are called *compensated or Hicksian demand* functions and are denoted $h(p,u)$. The minimal expenditure necessary to reach u is the *expenditure function*:

$$\sum_i p_i \cdot h_i(p,u) = e(p,u)$$

Remark 15 *Local non-satiation*

This assumption implies that $v(p,m)$ is strictly increasing in m. Thus we can derive the minimal expenditure necessary to reach u, $e(p,u)$, simply by inverting $v(p,m)$. It follows that $e(p,u)$ is strictly increasing in u.

Properties of the expenditure function

1. $e(p,u)$ is nondecreasing in p.
2. $e(p,u)$ is homogeneous of degree 1 in p.
3. $e(p,u)$ is concave in p.
4. $e(p,u)$ is continuous in p.
5. $\partial e(p,u)/\partial p_i = h_i(p,u)$.

These are the same properties that cost functions have!