Correspondence between a pair of dual problems

PRIMAL (DUAL)	DUAL (PRIMAL)
Maximise $Max\ Z$ $(Max\ W)$	Minimise $Min\ W$ $(Min\ Z)$
1 constraint	1 decision variable
i-th "$\le$" constraint	$y_i \ge 0$ $(x_i \ge 0)$
i-th "$\ge$" constraint	$y_i \le 0$ $(x_i \le 0)$
i-th "$=$" constraint	y_i free $(x_i$ free$)$
Right-hand-sides b_i $i=1,\dots,m$ $(c_j$ $j=1,\dots,n)$	Objective function's coefficients $W = b_1 y_1 + \dots + b_m y_m$ $(Z = c_1 x_1 + \dots + c_n x_n)$
Objective function's coefficients $Z = c_1 x_1 + \dots + c_n x_n$ $(W = b_1 y_1 + \dots + b_m y_m)$	Right-hand- sides c_j $j=1,\dots,n$ $(b_i$ $i=1,\dots,m)$
1 decision variable	1 constraint
$x_j \ge 0$ $(y_j \ge 0)$	j-th "$\ge$" constraint
$x_j \le 0$ $(y_j \le 0)$	j-th "$\le$"constraint
x_j free $(y_j$ free$)$	j-th "$=$"constraint
Technical coefficients' matrix A (A^T)	Technical coefficients' matrix A^T (A)

The **i-th shadow price** (y_i) represents the ratio of change in the objective function originated by an increase of the *i*-th right-hand-side.

Relation between the variables of a pair of dual problems:

Number of variables	Primal	Dual
n	Decision variables	Slack/surplus variables
m	Slack/surplus variables	Decision variables

Duality Properties

Property 1: **Symmetry** - The dual of the dual is the primal.

Property 2: **Week Duality theorem**

If $\mathbf{x}' = (x'_1, x'_2, \ldots, x'_n)$ is a feasible solution (FS) for the primal maximisation problem and $\mathbf{y}' = (y'_1, \ldots, y'_m)$ is a FS for the dual, then

$$Z' = c_1 x'_1 + c_2 x'_2 + \ldots + c_n x'_n \le b_1 y'_1 + b_2 y'_2 + \ldots + b_m y'_m = W' .$$

Property 3: If $\mathbf{x}^* = (x_1^*, \ldots, x_n^*)$ is a FS for the primal, $\mathbf{y}^* = (y_1^*, \ldots, y_m^*)$ is a FS for the dual and

$$Z^* = c_1 x_1^* + c_2 x_2^* + \ldots + c_n x_n^* = b_1 y_2^* + b_2 y_2^* + \ldots + b_m y_m^* = W^*$$

then, $\mathbf{x}^*$ e $\mathbf{y}^*$ are optimal solutions for the problems.

Property 4: **Strong Duality theorem**

Given a pair of dual problems, if one of them has optimum then the other also has optimum and both optimal values are equal, i.e., $Z^* = W^*$.

Property 5: Given a pair of dual problems, if one of them has feasible solutions and unbounded objective function (so no optimal solution), then the other has no feasible solutions.

Property 6: Given a pair of dual problems, if one of them has no feasible solutions, then the other has either no feasible solutions or an unbounded objective function.

Table: Primal/Dual solution

DUAL \ PRIMAL	**with FS**	**without FS**
with FS	Both problems have OS and $z^* = w^*$	**Primal without FS** **Dual unbounded**
without FS	**Primal unbounded** **Dual without FS**	**Primal without FS** **Dual without FS**

Property 7: The shadow prices are the optimal values for the decision variables of the dual.

Complementary relationships between a pair of dual solutions:

Number of variables	Primal variable	Dual variable
m	basic	non basic
$\ell - m$	non basic	basic