Exercises

1. Consider a metric space (X, d), and a mapping $\rho : X \times X \mapsto \mathbb{R}$ defined by
$$\rho(x,y) = \frac{d(x,y)}{1+d(x,y)}, \qquad x, y \in X.$$
Show that ρ is a metric and that it generates the same topology as d.

2. Consider a sequence $\{d_i, i \in \mathbb{N}\}$ of metrics defined on the same space X.
   (a) Show that
   $$\rho(x,y) = \sum_{i=1}^{+\infty} \frac{\min\{1, d_i(x,y)\}}{2^n}$$
   is a metric in X.
   (b) Show that the topology generated by ρ is stronger than any of the topologies generated by d_i, $i \in \mathbb{N}$.

3. Show that $L_\infty[0,1]$ provided with its natural topology is a Banach space.

4. Consider the sequence $\{f_n, n \in \mathbb{N}\}$ defined as
$$f_n(x) = \sum_{i=1}^{2^n} \left(\chi_{\left[\frac{2(i-1)}{2^{n+1}}, \frac{2i-1}{2^{n+1}}\right[}(x) - \chi_{\left[\frac{2i-1}{2^{n+1}}, \frac{2i}{2^{n+1}}\right[}(x) \right).$$
   (a) Show that for every $p \in [1, +\infty]$, f_n is a bounded sequence in $L_p[0,1]$.
   (b) Consider the map $\|\cdot\|_{1,\infty} : L_1[0,1] \mapsto \mathbb{R}$, defined as
   $$\|f\|_{1,\infty} = \sup_{x\in[0,1]} \left| \int_0^x f(t)dt \right|, \qquad \forall f \in L_1[0,1].$$
   Show that $\|\cdot\|_{1,\infty}$ is a norm in $L_1[0,1]$.

(c) Show that $\lim f_n = 0$ with respect to the norm $\|\cdot\|_{1,\infty}$.

(d) Use the results above to conclude that f_n does not admit any subsequence convergent in $L_p[0,1]$.

5. Consider an Hilbert space, $(H, \langle\cdot,\cdot\rangle)$ and a closed linear subspace $X \subset H$.

   (a) Show that for every $a \in H$ there exists one unique $b \in X$ such that
   $$\|b - a\| = \min_{x \in X} \|x - a\|.$$

   (b) Show that $\|b - a\| = \min_{x \in X} \|x - a\|$ if and only if
   $$\langle x - b, b - a\rangle = 0, \qquad \forall x \in X.$$
   Explain the geometric meaning of this equality.

6. Consider the functional
   $$J(x) = \int_0^1 x(t)(x(1-t) - 2t) + e^{x(t)^2} dt.$$

   (a) Show that J is Fréchet-differentiable in $L_\infty[0,1]$ but is not continuous in $L_2[0,1]$.

   (b) Show that J has one unique minimizer in $L_2[0,1]$.

   (c) Compute the minimizer of J (this can only be obtained as an implicit function of t).

7. Consider the functional
   $$J(x) = \int_0^1 1 + x'''(t)^2 dt.$$

   (a) Show that for any $x_0, x_1 \in \mathbb{R}$ the functional J admits one unique minimizer in $C_3[0,1]$ subject to the constraints $x(0) = x_0$, $x(1) = x_1$.

   (b) Find that minimizer.

8. Find the minimizer(s) of the functional
   $$J(x) = \int_0^1 \dot{x}(t)^2 - x(t) dt$$
   among all functions $x \in C_1[0,1]$ satisfying $x(0) = x(1) = 0$.

   Prove that the candidate solution is indeed a minimizer.

9. Consider the problem consisting of finding the shortest continuous curve lying in the unit radius cylinder

$$C = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\},$$

connecting to given points (x_0, y_0, z_0), $(x_1, y_1, z_1) \in C$.

   (a) Formulate the problem as an optimization problem;

   (b) Show that the functional to be minimized in the problem above is strictly convex. However, the problem admits infinitely many local minima.

   (c) Explain the behaviour described above.

10. Consider the optimal control problem consisting of finding minimizers for the functional

$$J(x, u) = \int_0^T 4x_1(t)^2 + u(t)^2 dt,$$

with dynamics

$$\dot{x}_1(t) = x_2(t), \qquad \dot{x}_2(t) = u(t),$$

and boundary conditions

$$x_1(0) = x_{1,0}, \quad x_2(0) = x_{2,0}, \quad x_1(T) = x_{1,T}, \quad x_2(T) = x_{2,T}$$

($T \in]0, +\infty[$ fixed). Admissible controls are measurable functions $u : [0, T] \mapsto [-1, 1]$.

   (a) Show that the set of points reachable from a given point $x_0 = (x_{1,0}, x_{2,0})$ in time T is a convex set with nonempty interior.

   (b) Show that the problem admits one unique solution for every boundary conditions $x_0 = (x_{1,0}, x_{2,0})$, $x_T = (x_{1,T}, x_{2,T})$ such that x_T can be reached from x_0 in time T.

   (c) Describe the optimal solution for generic boundary conditions (with x_T reachable from x_0).

11. Let $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^k \mapsto \mathbb{R}^n$, $L : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^k \mapsto \mathbb{R}$, $g : \mathbb{R}^n \mapsto \mathbb{R}$ be smooth functions.

    Formulate the Pontryagin maximum principle for the following problems:

(a) Minimize

$$J(x,u) = \int_0^T L(t, x(t), u(t))dt + g(x(T)),$$

subject to:

$$\dot{x}(t) = f(t, x(t), u(t)), \qquad x(0) = x_0, \qquad x(T) \in X_T,$$
$$u \in L_\infty[0,T],$$

with $T \in]0, +\infty[$ fixed.

(b) Find the control $u \in L^k_\infty[0, +\infty[$ that drives the trajectory of the system

$$\dot{x}(t) = f(t, x(t), u(t)), \qquad x(0) = x_0$$

to the origin in the shortest possible time interval.

12. Let $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^k \mapsto \mathbb{R}^n$, $L : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^k \mapsto \mathbb{R}$ be smooth functions.

Let $\mathcal{M}$ denote a set of bounded measurable functions with domain $[0,T]$ and range $\mathbb{R}^k$ satisfying the following conditions:

$\mathcal{M}$ contains all constant functions;

For every $u_1, u_2 \in \mathcal{M}$ and every $\theta \in]0,T[$, the function

$$u(t) = \begin{cases} u_1(t), & \text{if } t \in [0, \theta[; \\ u_2(t), & \text{if } t \in [\theta, T], \end{cases}$$

is in $\mathcal{M}$.

Consider the optimal control problem

$$\text{Minimize } J(x,u) = \int_0^T L(t, x(t), u(t))dt,$$

subject to:

$$\dot{x}(t) = f(t, x(t), u(t)), \qquad x(0) = x_0, \qquad x(T) = x_T, \qquad u \in \mathcal{M}.$$

Show that any solution of the problem above must satisfy the Pontryagin maximum principle.

13. Consider the optimal control problem:

$$\text{minimize } J(x,u) = \int_0^1 u(t)^2 dt$$
$$\text{subject to} : \dot{x}_1(t) = x_3(t), \quad \dot{x}_2(t) = x_1(t), \quad \dot{x}_3(t) = u(t),$$

with boundary conditions

$$x_i(0) = x_{i,0}, \quad i = 1,2,3, \qquad x_i(1) = x_{i,1}, \quad i = 1,2,3.$$

(a) Show that for each boundary condition there is one unique curve $x(t) = (x_1(t), x_2(t), x_3(t))$, $t \in [0,1]$ satisfying the Pontryagin maximum principle.

(b) Show that if the dynamics is changed to

$$\dot{x}_1(t) = x_2x_3(t), \quad \dot{x}_2(t) = x_1x_3(t), \quad \dot{x}_3(t) = u(t),$$

then **all** trajectories of the new system satisfy the Pontryagin maximum principle.

(c) Explain the phenomenon observed above.

14. An investor can chose between investment and consumption. His current wealth at time t is denoted by $w(t)$. This evolves in time according to the dynamics

$$\dot{w}(t) = a + rw(t) - c(t),$$

Where $c(t)$ denotes the rate of consumption at moment t. Consumption is valued by a logarithmic utility function

$$u(c) = \log c.$$

Assuming that his subjective discount rate is a constant $\beta > 0$, the investor seeks to maximize the total utility

$$U(w,c) = \int_0^T e^{-\beta t} \log c(t) dt + e^{-\beta T} w(T),$$

given an initial wealth $w(0) = w_0$.

(a) Find the optimal choice of consumption $c : [0,T] \mapsto [0,+\infty[$.

(b) If an additional constraint $w(T) \geq 0$ is added, how the optimal choice will change?

15. Consider a market with two suppliers, A and B. Let $x_A(t)$, $x_B(t)$ denote the number of costumers buying form company A and company B respectively, at time t. These numbers evolve according to the dynamics

$$\dot{x}_A(t) = x_B(t)u(t), \qquad \dot{x}_B(t) = -x_B(t)u(t),$$

where u denotes the publicity effort of company A and is constrained to the interval $[0, 1]$. This company seeks to maximize the number of costumers over a time period $[0, T]$, deduced of the publicity costs in the same period. That is, it seeks to minimize the functional

$$J(x, u) = \int_0^T x_B(t) + u(t)dt.$$

(a) Show the problem admits a solution.

(b) Solve the problem for arbitrary initial conditions.

16. Consider an individual having some amount of capital. Part of his wealth is kept in a bank account from where he pays current expenses. The remaining is invested in a risk-less financial asset (e.g., bonds).

We model this situation using the following variables:

$x(t)$ is the individual's bank account balance at time t (can be positive or negative);

$y(t)$ is the value of financial assets that he owns at time t (negative $y(t)$ means a short position at time t);

$d(t)$ is the instantaneous cash-flow due to current expenses at time t. This is assumed to be known a-priori for the period $[0, T]$;

$u(t)$ is the instantaneous rate at which the individual is selling ($u > 0$) or buying ($u < 0$) assets at time t. This rate has a-priori bounds: $-U_1 \leq u(t) \leq U_2$, $\forall t \geq 0$;

$r_1(t)$, $r_2(t)$ are the interest rates of the bank account and the financial assets, respectively. These are assumed to be known a-priori for the period $[0, T]$;

$\alpha > 0$ is the transaction charged by each unit transaction (buy or sell) of the financial asset;

The evolution of these variables is described by the dynamics

$$\dot{x}(t) = r_1(t)x(t) - d(t) + u(t) - \alpha|u(t)|, \qquad \dot{y}(t) = r_2(t)y(t) - u(t).$$

The individual wants to plan his investment strategy for the period $[0, T]$ (i.e., to chose $u : [0, T] \mapsto [-U_1, U_2]$), with the aim of maximizing his total wealth at the end of the period (i.e., to maximize $x(T)+y(T)$).

Solve this problem.

17. Consider the optimal control problem

$$J(u) = \int_0^T \ln u(t)dt \to \max$$
$$\dot{x}(t) = ax(t) - u(t), \qquad x(0) = x_0, x(T) = x_T, \qquad u > 0.$$

(a) Show that

$$S(t, x) = (T - t) \ln \left(\frac{xe^{-at} - x_T e^{-aT}}{T - t} \right) + \frac{a}{2}(T^2 - t^2)$$

solves the Hamilton-Jacobi-Bellman equation associated to this problem.

(b) Solve the problem.

18. Let $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^k \mapsto \mathbb{R}^n$, $L_1, L_2 : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^k \mapsto \mathbb{R}$ be smooth functions. Suppose there is a C_1-function, $S : [0, T] \times \mathbb{R}^n \mapsto \mathbb{R}$ satisfying

$$L_1(t, x, u) - L_2(t, x, u) = \frac{\partial S}{\partial t}(t, x) + \frac{\partial S}{\partial x}(t, x)f(t, x, u),$$
$$\forall (t, x, u) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^k.$$

Show that any minimizer of the functional

$$J_1(x, u) = \int_0^T L_1(t, x(t), u(t))dt,$$

subject to:

$$\dot{x}(t) = f(t, x(t), u(t)) \quad a.e.\ t \in [0, T], \qquad u \in L^k_\infty[0, T]$$
$$x(0) = x_0, \qquad x(T) = x_T,$$

is also a minimizer of the functional

$$J_2(x, u) = \int_0^T L_2(t, x(t), u(t))dt,$$

subject to the same constraints.

19. Consider a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ and a Brownian motion $W_t : \Omega \mapsto \mathbb{R}$, adapted to $\{\mathcal{F}_t\}$. Let $\mathcal{U}$ denote the set of all $\{\mathcal{F}_t\}$-adapted processes, $u_t : \Omega \mapsto \mathbb{R}$ such that

$$E\left[\int_0^T |u_t| dt\right] < +\infty$$

holds for every $T \in]0, +\infty[$. For each $u_t \in \mathcal{U}$ we consider the $\{\mathcal{F}_t\}$-adapted processes X_t, Y_t satisfying the dynamics

$$dX_t = Y_t dt, \qquad dY_t = u_t dt + dW_t.$$

We introduce the functional

$$J(t,x,y,u) = E\left[X_T^2 + \int_0^T u_t^2 dt \,\middle|\, X_t = x, Y_t = y\right], \\ (t,x,y) \in [0,T[\times\mathbb{R}^2, \quad u \in \mathcal{U}.$$

Find the processes u (depending on (t,x,y)) such that

$$J(t,x,y,u) = \min_{v\in\mathcal{U}} J(t,x,y,v).$$

20. Consider an individual whose wealth can be placed in a bank account, invested in stock or used for consumption. We use the following notation:

$W_t \in \mathbb{R}$, the total wealth of the individual at time t;

$C_t \in [0, +\infty[$, the consumption rate at time t;

$Q_t \in \mathbb{R}$, the fraction of total wealth invested in stock at time t (For $W_t > 0$, $Q_t < 0$ means a short position, while $Q_t > 1$ means the investor is holding stock using borrowed money);

$(1 - Q_t)$, the fraction of total wealth kept in the bank account;

Consider a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ and let $B_t : \Omega \mapsto \mathbb{R}$ denote a Bownian motion adapted to $\{\mathcal{F}_t\}$.

We assume that the investor's wealth is an $\{\mathcal{F}_t\}$-adapted stochastic process with dynamics

$$\begin{aligned} dW_t =& \alpha Q_t W_t dt + \sigma Q_t W_t dB_t + r(1-Q_t)W_t dt - C_t dt = \\ =& (\alpha - r)Q_t W_t dt + (rW_t - C_t)dt + \sigma Q_t W_t dB_t, \end{aligned}$$

where α is the average rate of return in the stock market, $\sigma > 0$ is the stock volatility and r is the interest rate of the bank account.

Let $T = \inf\{t \geq 0 : W_t \leq 0\}$. T is a stopping time when the individual is bankrupt and can no longer invest.

Consumption is valued by a logarithmic utility function and discounted at a subjective rate $\rho > 0$. So we assume the investor seeks to maximize the functional

$$J(w, C) = E\left[\int_0^T C_t^{-\rho t} dt \middle| W_0 = w\right].$$

Find the optimal consumption plan given initial wealth $W_0 = w$.

21. An investor deals with two assets. Asset B is risk-free and its price evolves according to the dynamics

$$dB_t = r_1 B_t dt.$$

S denotes the price of a risky asset whith dynamics

$$dS_t = r_2 S_t dt + v S_t dW_t,$$

where W is a Brownian motion defined in some appropriate probability space. Π_t and $(1 - \Pi_t)$ denote the proportion of the investor's wealth invested in asset B and asset S, respectively. r_1, r_2, v are known positive constants. We presume that the investor can take any amount of debt and that he can take short positions (i.e., Π can take any real value).

At any moment the investor can adjust his portfolio. However, conversion of one asset into another is neither costless nor instantaneous. Though there is no lower bound for the amount of time required to trade any particular amount of assets, faster transactions will be performed in less favorable conditions, and hence carry some penalty. This is included in the model through the equations:

$$\begin{aligned} d\Pi_t &= C_t dt, \\ dX_t &= \Pi_t \frac{X_t}{B_t} dB_t + (1 - \Pi_t) \frac{X_t}{S_t} dS_t - \alpha C_t^2 dt, \end{aligned}$$

where X_t denotes the total wealth of the investor, $C_t \in \mathbb{R}$ is the speed of transaction and $\alpha > 0$ is a constant.

Let $\mathcal{C}$ be the set of all processes adapted to the filtration generated by the Brownian motion W, such that

$$E\left[\int_0^T C_t^2 dt\right] < +\infty.$$

Assume that the PDE

$$\frac{\partial V}{\partial t} + \frac{1}{4\alpha}\left(\frac{\partial V}{\partial \pi}\right)^2\left(\frac{\partial V}{\partial x}\right)^{-1} + ((r_1 - r_2)\pi + r_2)x\frac{\partial V}{\partial x} + \\ + \frac{1}{2}v^2x^2(1-\pi)^2\frac{\partial^2 V}{\partial \pi^2} = 0$$

admits a solution in $\mathbb{R}^2 \times [0, T[$ that can be extended by continuity into $\mathbb{R}^2 \times [0, T]$ and

$$V(T, \pi, x) = -x, \qquad \forall \pi, x \in \mathbb{R}.$$

Show that, for any initial position there exists an investment strategy that maximizes the expected value of the total wealth at the final moment T (fixed). Show that such a strategy is specified by a function $C \in \mathcal{C}$ satisfying

$$C_t = \frac{1}{2\alpha}\frac{\partial V}{\partial \pi}(t, \Pi_t, X_t)\left(\frac{\partial V}{\partial x}(t, \Pi_t, X_t)\right)^{-1}.$$

22. The one-dimensional equation

$$u(x) + \frac{\dot{u}(x)^2}{2} = 0, \qquad x \in \mathbb{R}$$

admits the classical solutions $u_1 \equiv 0$ and and $u_2(x) = \frac{(x-x_0)^2}{2}$ (for any $x_0 \in \mathbb{R}$).

Prove that u_1 is the unique continuous viscosity solution.

23. Consider the partial differential equation:

$$\frac{\partial u}{\partial t} + \left(\frac{\partial u}{\partial x}\right)^2 = 0, \qquad (t, x) \in]0, +\infty[\times\mathbb{R},$$
$$u(0, x) = 0, \qquad \forall x \in \mathbb{R}.$$

(a) Show that $u_1(t, x) \equiv 0$ is a viscosity solution.

(b) Show that

$$u_2(t,x) = \begin{cases} 0, & \text{for } |x| \geq t, \\ x - t, & \text{for } 0 \leq x < t, \\ -(x+t), & \text{for } -t < x \leq 0, \end{cases}$$

is not a viscosity solution.

24. Consider the control system

$$\dot{x}_1(t) = x_2(t), \quad \dot{x}_2(t) = u(t), \qquad |u(t)| \leq 1,\ t \in [0, +\infty[.$$

For each measurable $u : [0, +\infty[\mapsto [-1, 1]$ and each $x_0 \in \mathbb{R}^2$, let x_{u,x_0} denote the corresponding solution of the equation above with initial value $x(0) = x^0$, and let

$$T_{u,x_0} = \inf\{t \geq 0 : x_{u,x_0}(t) = 0\}$$

(with $T_{u,x_0} = +\infty$ if $x_{u,x_0}(t) \neq 0\ \forall t \in [0, +\infty[$).

We wish to find the control u minimizing T_{u,x_0}.

(a) Show that the Hamilton-Jacobi-Bellman equation for this problem is

$$\sup_{u \in [-1,1]} \left(-x_2 \frac{\partial V}{\partial x_1} - u \frac{\partial V}{\partial x_2} - 1 \right) = 0, \qquad (t, x_1, x_2) \in [0, +\infty[\times \mathbb{R}^2,$$
$$V(t, 0, 0) = 0, \qquad \forall t \geq 0.$$

(b) Find a viscosity solution for this equation and find the optimal feedback.