Instituto Superior de Economia e Gestão

UNIVERSIDADE TÉCNICA DE LISBOA

Master in Actuarial Science

Risk Theory

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Lecturer: Maria de Lourdes Centeno

Solution

1. (a) Let N be the number of disabilities per year.

   $E[N]=\lambda=1000\times\frac{1}{100}=10$

   The probability function of the disability period, Y, is

   $$g(y)=\frac{1}{30},\ y=1,2,...,30.$$

   $E[Y]=\sum_{y=1}^{30}yg(y)=\frac{1}{30}\times\frac{1+30}{2}\times30=\frac{31}{2}=\frac{31}{2}$

   $E[Y^2]=\sum_{y=1}^{30}y^2g(y)=\frac{1}{30}\times\frac{30\times(1+30)\times(1+2\times30)}{6}=\frac{1891}{6}=\frac{1891}{6}$

   $E[Y^3]=\sum_{y=1}^{30}y^3g(y)=\frac{1}{30}\times\frac{30^2\times(1+30)^2}{4}=\frac{14\,415}{2}=\frac{14\,415}{2}$

   Let $X=100Y$ be the individual claim amount and S the aggregate claim amount (compound Poisson), then

   $\mu_S=E[S]=E[N]E[X]=10\times100\times\frac{31}{2}=15\,500=15\,500$

   $Var[S]=E[N]E[X^2]=10\times100^2\times\frac{1891}{6}=\frac{94\,550\,000}{3}$, i.e. $\sigma_S=\sqrt{\frac{94\,550\,000}{3}}=5614.0$

   $E[(S-\mu_S)^3]=E[N]E[X^3]=10\times100^3\times\frac{14\,415}{2}=7.207\,5\times10^{10}$, hence $\gamma_S=\frac{7.207\,5\times10^{10}}{\left(\frac{94\,550\,000}{3}\right)^{3/2}}=0.407\,36$

   (b) Using the Normal Power approximation we have

   $\Pr(S>22500)=\Pr\left(Z>\frac{22500-15\,500}{5614.0}\right)=$

   $=\Pr\left(Z>1.246\,9\right)\simeq1-\Phi\left(-\frac{3}{\gamma_S}+\sqrt{\frac{9}{\gamma_S^2}+1+\frac{6\times1.246\,9}{\gamma_S}}\right)=$

   $=1-\Phi\left(-\frac{3}{0.407\,36}+\sqrt{\frac{9}{0.407\,36^2}+1+\frac{6\times1.246\,9}{0.407\,36}}\right)=1-\Phi(1.214\,6)=0.112259344$

   The student could also use the translated gamma approximation.

(c) Let $S^* = S\ /100$ (and note that $X/100 = Y$)
We want to calculate $\Pr(S^* > 4) = 1 - f((0) + f(1) + f(2) + f(3))$
Using Panjer's recursion formulae

$$\begin{aligned} f(i) &= \frac{\lambda}{i}\sum_{j=1}^{i} jg(j)f(i-j), \\ f(0) &= \exp(-\lambda) \end{aligned}$$

we get
$f(0) = \exp(-10) = e^{-10} = 4.5400 \times 10^{-5}$
$f(1) = 10 \times \frac{1}{30} \times 4.5400 \times 10^{-5} = 1.5133 \times 10^{-5}$
$f(2) = \frac{10}{2} \times (1 \times \frac{1}{30} \times 1.5133 \times 10^{-5} + 2 \times \frac{1}{30} \times 4.5400 \times 10^{-5}) = 1.7656 \times 10^{-5}$
$f(3) = \frac{10}{3} \times (1 \times \frac{1}{30} \times 1.7656 \times 10^{-5} + 2 \times \frac{1}{30} \times 1.5133 \times 10^{-5} + 3 \times \frac{1}{30} \times 4.5400 \times 10^{-5}) = 2.0458 \times 10^{-5}$
$f(4) = \frac{10}{4} \times (1 \times \frac{1}{30} \times 2.0458 \times 10^{-5} + 2 \times \frac{1}{30} \times 1.7656 \times 10^{-5} + 3 \times \frac{1}{30} \times 1.5133 \times 10^{-5} + 4 \times \frac{1}{30} \times 4.5400 \times 10^{-5}) = 2.3564 \times 10^{-5}$
$F(4) = 4.5400 \times 10^{-5} + 1.5133 \times 10^{-5} + 1.7656 \times 10^{-5} + 2.0458 \times 10^{-5} + 2.3564 \times 10^{-5} = 1.2221 \times 10^{-4}$
The probability that the aggregate claims are greater 400m.u. is $1 - 1.2221 \times 10^{-4} = 0.99988$

2. $X \sim Gamma(\alpha, \beta)$
$N \sim Po(\lambda)$
$E[X] = \frac{\alpha}{\beta} = 4$
$Var[X] = \frac{\alpha}{\beta^2} = 8$
then $\alpha = 2$, $\beta = \frac{1}{2}$.
$\theta = 0.3$

(a) $M_X(r) = \left(\frac{\frac{1}{2}}{\frac{1}{2}-r}\right)^2 = (1-2r)^{-2}$
$1 + (1+\theta)a_1 r - M_X(r) = 1 + 1.3 \times 4r - (1-2r)^{-2} = 1 + 5.2r - (1-2r)^{-2}$
$\int_0^{+\infty} \exp(ur)[-\psi'(u)]du = \frac{\theta}{1+\theta}\frac{M_X(r)-1}{1+(1+\theta)a_1 r - M_X(r)} = \frac{0.3}{1.3}\frac{(1-2r)^{-2}-1}{1+5.2r-(1-2r)^{-2}} =$
$= \frac{0.3}{1.3}\frac{1-(1-2r)^2}{(1-2r)^2+5.2r(1-2r)^2-1} = \frac{0.3}{1.3}\frac{4r-4r^2}{1.2r-16.8r^2+20.8r^3} = \frac{0.3}{1.3}\frac{4-4r}{1.2-16.8r+20.8r^2} =$
$= \frac{0.3}{1.3}\frac{4-4r}{20.8(0.728499-r)(0.079193-r)} = \frac{3}{270.4}\frac{4-4r}{(0.728499-r)(0.079193-r)} =$
$= \frac{3}{270.4}\frac{4-4r}{(0.728499-r)(0.079193-r)} = \frac{0.728499 \times C_1}{0.728499-r} + \frac{0.079193 \times C_2}{0.079193-r}$ with $C_1 = -0.025472252$ and $C_2 = 0.794707234$
Hence $\psi(u) = 0.794707\exp(-0.079193u) - 0.025472\exp(-0.728499u)$.

(b) $R = 0.079193$, since R must be smaller then $\frac{1}{2}$. Then $\psi(u) \leq \exp(-0.079193u)$
$\psi(50) \leq \exp(-0.079193 \times 50) = 1.9070 \times 10^{-2}$
$\psi(50) = 0.794707\exp(-0.079193 \times 50) - 0.025472\exp(-0.728499 \times 50) = 1.5155 \times 10^{-2}$.

3.

$$\begin{cases} \frac{\beta}{\alpha-1} = 1 \\ \frac{\alpha\beta^2}{(\alpha-2)(\alpha-1)^2} = 3 \end{cases} \text{, Solution is : } [\alpha = 3, \beta = 2]$$

(a) $\Pr[X - 0.05 \leq x | X > 0.05] = \frac{\Pr[0.05 < X \leq x + 0.05]}{\Pr[X > 0.05]} = \frac{1 - \left(\frac{2}{2.05+x}\right)^3 - 1 + \left(\frac{2}{2..05}\right)^3}{\left(\frac{2}{2.05}\right)^3} = \frac{\left(\frac{2}{2.05}\right)^3 - \left(\frac{2}{2.05+x}\right)^3}{\left(\frac{2}{2.05}\right)^3} = 1 - \left(\frac{2.05}{2.05+x}\right)^3$, which is a Pareto with parameters 3 and 2.05.

(b) $E[Z|Z > 0]$ is the mean of the Pareto (3, 2.05). Hence $E[Z|Z > 0] = \frac{2.05}{3-1} = 1.025$.

$E[Z] = \int_{0.05}^{\infty} \left(\frac{2}{2+x}\right)^3 dx = 0.95181$

$E[Y] = E[X] - E[Z] = 1 - 0.95181 = 0.04819$

(c) c - gross (of reinsurance) premium

$E[N] = 100$

$P(a)$ - reinsurance premium

aZ - individual retained claim amount

$(1 - a)Z$ - individual ceded claim amount

W - net profit

$c = 1.1 \times 100 \times E[Z] = 1.1 \times 100 \times 0.95181 = 104.70$

$P(a) = 1.25 \times 100 \times (1 - a)E[Z] = 1.25 \times 100 \times 0.95181 \times (1 - a) =$

$= 118.98(1 - a)$

$E[W] = c - P(a) - aE[N]E[Z] = 104.70 - 118.98(1 - a) - 100 \times 0.95181a$

$= 23.799a - 14.28$

(d) The adjustment coefficient of the retained risk will not exist for any $0 < a < 1$, because the retained claim will still be a Pareto and the Pareto does not have moment generating function.

4. Let $d = \text{VaR}_p(X)$

(a) $F_X(d) = p$

$1 - \left(\frac{\beta}{\beta+d}\right)^{\alpha} = p$

$d = \beta\left[(1 - p)^{-\frac{1}{\alpha}} - 1\right]$

Hence $\text{VaR}_p(X) = \beta\left[(1 - p)^{-\frac{1}{\alpha}} - 1\right]$.

(b) $\text{CTE}_p(X) = E[X|X > d] = \int_0^{+\infty} \Pr(X > x|X > d)\, dx$

But $\Pr(X > x|X > d) = \begin{cases} \frac{\Pr(X>x)}{\Pr(X>d)} = \left(\frac{\beta+d}{\beta+x}\right)^{\alpha} & \text{if } x > d \\ 1 & \text{if } x \leq d \end{cases}$

Then $\text{CTE}_p(X) = \int_0^d 1dx + \int_d^{+\infty} \left(\frac{\beta+d}{\beta+x}\right)^{\alpha} dx = \beta\left[(1 - p)^{-\frac{1}{\alpha}} - 1\right] + \frac{\beta(1-p)^{-\frac{1}{\alpha}}}{\alpha-1} = \beta\left[\frac{\alpha}{\alpha-1}(1 - p)^{-\frac{1}{\alpha}} - 1\right]$

The solution could also be found noticing that for continuous distributions the conditional tail expectation is equal to the Tail- Value at- Risk, i.e. that $\text{CTE}_p(X) = \text{TVaR}_p(X) = \frac{1}{1-p}\int_p^1 \text{VaR}_t(X)dt = \frac{1}{1-p}\int_p^1 \beta\left[(1-t)^{-\frac{1}{\alpha}} - 1\right]dt = \beta\left[\frac{\alpha}{\alpha-1}(1-p)^{-\frac{1}{\alpha}} - 1\right]$

5. The discretized distribution is

$$\begin{cases} g_0 = 1 - \exp(-h/2) \\ g_j = \exp(-h(j-1/2)) - \exp(-h(j+1/2)), \ j = 1, 2, ... \end{cases}$$

Which can be written as

$$\begin{cases} g_0 = 1 - \exp(-h/2) \\ g_j = \exp(-hj)\left(\exp(h/2) - \exp(-h/2)\right), \ j = 1, 2, ... \end{cases}$$

which can be regarded as a modification at zero of the geometric distribution $f_j = pq^j$, $j = 0, 1, ...$i.e.

$g_j = \frac{1-g_0}{1-f_0} f_j$, $j = 1, 2, ...$ with $g_0 = 1 - \exp(-h/2)$ and $q = \exp(-h)$.