Statistics for Business and Economics

7th Edition

Chapter 8

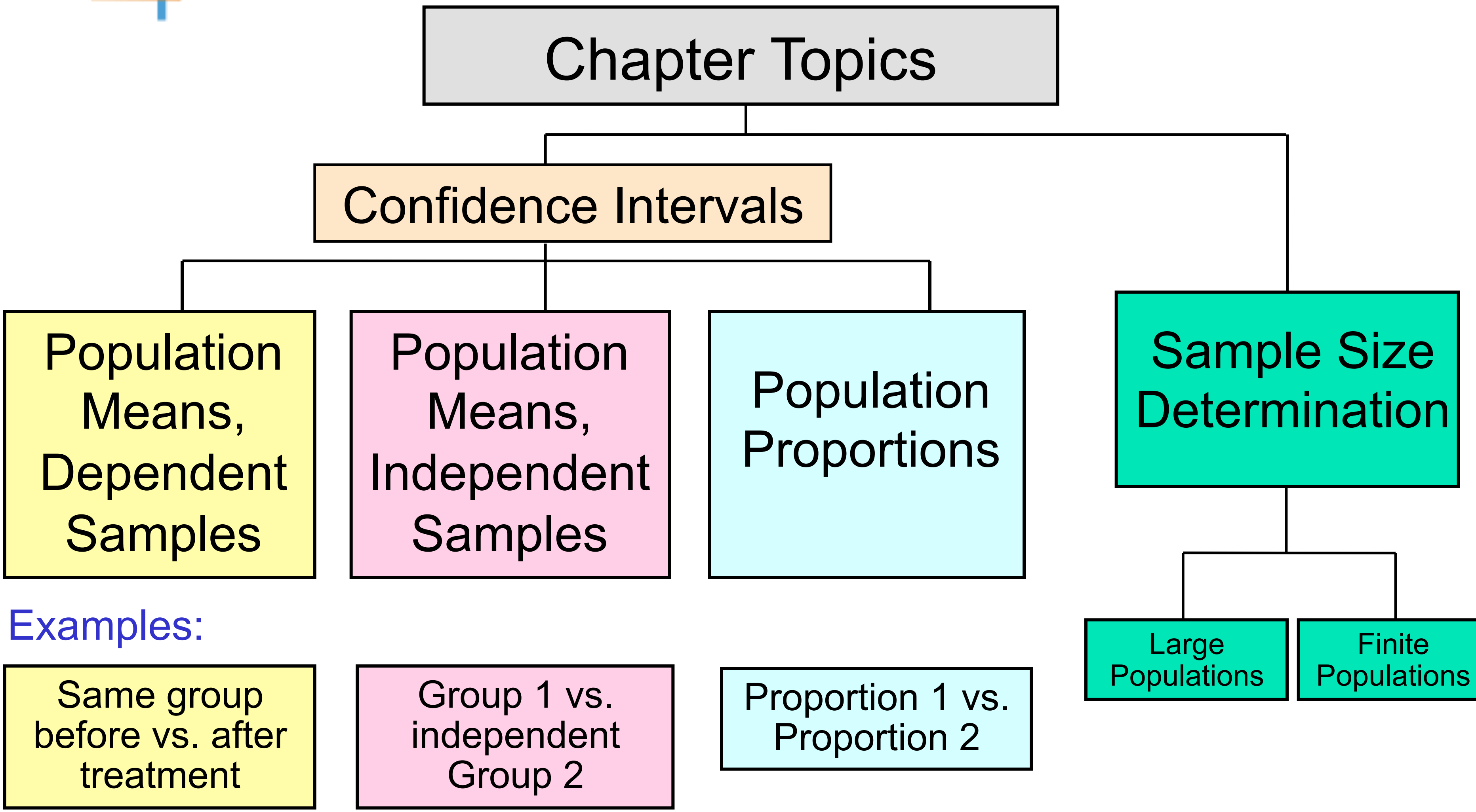
Estimation: Additional Topics

Chapter Goals

After completing this chapter, you should be able to:

- Form confidence intervals for the difference between two means from dependent samples
- Form confidence intervals for the difference between two independent population means (standard deviations known or unknown)
- Compute confidence interval limits for the difference between two independent population proportions
- Determine the required sample size to estimate a mean or proportion within a specified margin of error

Estimation: Additional Topics



8.1

Dependent Samples

Dependent samples

Tests Means of 2 Related Populations

- Paired or matched samples
- Repeated measures (before/after)
- Use difference between paired values:

$$\mathbf{d_i = x_i - y_i}$$

- Eliminates Variation Among Subjects
- Assumptions:
  - Both Populations Are Normally Distributed

Mean Difference

Dependent samples

The i^{th} paired difference is d_i , where

$$d_i = x_i - y_i$$

The point estimate for the population mean paired difference is $\bar{d}$:

$$\bar{d} = \frac{\sum_{i=1}^{n} d_i}{n}$$

The sample standard deviation is:

$$S_d = \sqrt{\frac{\sum_{i=1}^{n} (d_i - \bar{d})^2}{n-1}}$$

n is the number of matched pairs in the sample

Confidence Interval for Mean Difference

Dependent samples

The confidence interval for difference between population means, μ_d , is

$$\bar{d} - t_{n-1,\alpha/2} \frac{S_d}{\sqrt{n}} < \mu_d < \bar{d} + t_{n-1,\alpha/2} \frac{S_d}{\sqrt{n}}$$

Where

n = the sample size

(number of matched pairs in the paired sample)

Confidence Interval for Mean Difference

(continued)

Dependent samples

- The margin of error is

$$ME = t_{n-1,\alpha/2} \frac{s_d}{\sqrt{n}}$$

- $t_{n-1,\alpha/2}$ is the value from the Student's t distribution with (n – 1) degrees of freedom for which

$$P(t_{n-1} > t_{n-1,\alpha/2}) = \frac{\alpha}{2}$$

Paired Samples Example

Dependent samples

- Six people sign up for a weight loss program. You collect the following data:

Person	Weight: Before (x)	After (y)	Difference, d_i
1	136	125	11
2	205	195	10
3	157	150	7
4	138	140	- 2
5	175	165	10
6	166	160	6
			42

$$\bar{d} = \frac{\sum d_i}{n}$$

$$= 7.0$$

$$S_d = \sqrt{\frac{\sum (d_i - \bar{d})^2}{n-1}}$$

$$= 4.82$$

Paired Samples Example

(continued)

Dependent samples

- For a 95% confidence level, the appropriate t value is $t_{n-1,\alpha/2} = t_{5,.025} = 2.571$
- The 95% confidence interval for the difference between means, μ_d, is

$$\bar{d} - t_{n-1,\alpha/2}\frac{S_d}{\sqrt{n}} < \mu_d < \bar{d} + t_{n-1,\alpha/2}\frac{S_d}{\sqrt{n}}$$

$$7 - (2.571)\frac{4.82}{\sqrt{6}} < \mu_d < 7 + (2.571)\frac{4.82}{\sqrt{6}}$$

$$-1.94 < \mu_d < 12.06$$

Since this interval contains zero, we cannot be 95% confident, given this limited data, that the weight loss program helps people lose weight

Difference Between Two Means: Independent Samples

8.2

Population means, independent samples

Goal: Form a confidence interval for the difference between two population means, $\mu_x - \mu_y$

- Different data sources
  - Unrelated
  - Independent
    - Sample selected from one population has no effect on the sample selected from the other population
- The point estimate is the difference between the two sample means:

$$\bar{x} - \bar{y}$$

Difference Between Two Means: Independent Samples

(continued)

Population means, independent samples

- σ_x^2 and σ_y^2 known → Confidence interval uses $z_{\alpha/2}$
- σ_x^2 and σ_y^2 unknown
  - σ_x^2 and σ_y^2 assumed equal
  - σ_x^2 and σ_y^2 assumed unequal

  → Confidence interval uses a value from the Student's **t** distribution

σ_x^2 and σ_y^2 Known

Population means, independent samples

- σ_x^2 and σ_y^2 known *
- σ_x^2 and σ_y^2 unknown

Assumptions:

- Samples are randomly and independently drawn
- both population distributions are normal
- Population variances are known

σ_x^2 and σ_y^2 Known

(continued)

Population means, independent samples

- σ_x^2 and σ_y^2 known *
- σ_x^2 and σ_y^2 unknown

When σ_x and σ_y are known and both populations are normal, the variance of $\overline{X} - \overline{Y}$ is

$$\sigma^2_{\overline{X}-\overline{Y}} = \frac{\sigma_x^{\ 2}}{n_x} + \frac{\sigma_y^{\ 2}}{n_y}$$

…and the random variable

$$Z = \frac{(\overline{x} - \overline{y}) - (\mu_X - \mu_Y)}{\sqrt{\dfrac{\sigma_x^2}{n_X} + \dfrac{\sigma_y^2}{n_Y}}}$$

has a standard normal distribution

Confidence Interval, σ_x^2 and σ_y^2 Known

Population means, independent samples

- σ_x^2 and σ_y^2 known
- σ_x^2 and σ_y^2 unknown

* The confidence interval for $\mu_x - \mu_y$ is:

$$(\bar{x}-\bar{y}) - z_{\alpha/2}\sqrt{\frac{\sigma_X^2}{n_x}+\frac{\sigma_Y^2}{n_y}} < \mu_X - \mu_Y < (\bar{x}-\bar{y}) + z_{\alpha/2}\sqrt{\frac{\sigma_X^2}{n_x}+\frac{\sigma_Y^2}{n_y}}$$

σ_x^2 and σ_y^2 Unknown, Assumed Equal

- Population means, independent samples
  - σ_x^2 and σ_y^2 known
  - σ_x^2 and σ_y^2 unknown
    - σ_x^2 and σ_y^2 assumed equal *
    - σ_x^2 and σ_y^2 assumed unequal

Assumptions:

- Samples are randomly and independently drawn
- Populations are normally distributed
- Population variances are unknown but assumed equal

σ_x^2 and σ_y^2 Unknown, Assumed Equal

(continued)

- Population means, independent samples
  - σ_x^2 and σ_y^2 known
  - σ_x^2 and σ_y^2 unknown
    - σ_x^2 and σ_y^2 assumed equal *
    - σ_x^2 and σ_y^2 assumed unequal

Forming interval estimates:

- The population variances are assumed equal, so use the two sample standard deviations and pool them to estimate σ
- use a t value with $(n_x + n_y - 2)$ degrees of freedom

σ_x^2 and σ_y^2 Unknown, Assumed Equal

(continued)

- Population means, independent samples
  - σ_x^2 and σ_y^2 known
  - σ_x^2 and σ_y^2 unknown
    - σ_x^2 and σ_y^2 assumed equal *
    - σ_x^2 and σ_y^2 assumed unequal

The pooled variance is

$$s_p^2 = \frac{(n_x - 1)s_x^2 + (n_y - 1)s_y^2}{n_x + n_y - 2}$$

Confidence Interval, σ_x^2 and σ_y^2 Unknown, Equal

- σ_x^2 and σ_y^2 unknown
  - σ_x^2 and σ_y^2 assumed equal *
  - σ_x^2 and σ_y^2 assumed unequal

The confidence interval for $\mu_1 - \mu_2$ is:

$$(\bar{x}-\bar{y}) - t_{n_x+n_y-2,\alpha/2}\sqrt{\frac{s_p^2}{n_x}+\frac{s_p^2}{n_y}} < \mu_X - \mu_Y < (\bar{x}-\bar{y}) + t_{n_x+n_y-2,\alpha/2}\sqrt{\frac{s_p^2}{n_x}+\frac{s_p^2}{n_y}}$$

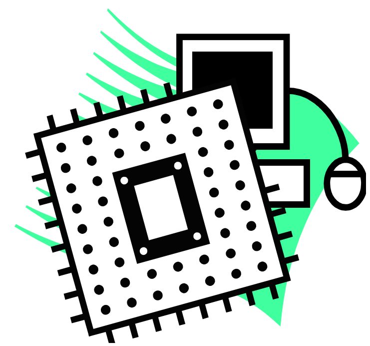
Where

$$s_p^2 = \frac{(n_x-1)s_x^2 + (n_y-1)s_y^2}{n_x+n_y-2}$$

Pooled Variance Example

You are testing two computer processors for speed. Form a confidence interval for the difference in CPU speed. You collect the following speed data (in Mhz):

	CPU_x	CPU_y
Number Tested	**17**	**14**
Sample mean	**3004**	**2538**
Sample std dev	**74**	**56**



Assume both populations are normal with equal variances, and use 95% confidence

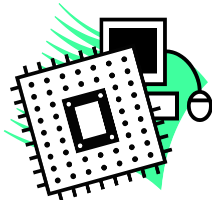
Calculating the Pooled Variance

The pooled variance is:

$$S_p^2 = \frac{(n_x - 1)S_x^{\,2} + (n_y - 1)S_y^{\,2}}{(n_x - 1) + (n_y - 1)} = \frac{(17 - 1)74^2 + (14 - 1)56^2}{(17 - 1) + (14 - 1)} = \boxed{4427.03}$$

The t value for a 95% confidence interval is:

$$\boxed{t_{n_x + n_y - 2,\, \alpha/2} = t_{29,\, 0.025} = 2.045}$$



Calculating the Confidence Limits

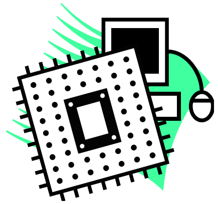
- The 95% confidence interval is

$$(\bar{x}-\bar{y})-t_{n_x+n_y-2,\alpha/2}\sqrt{\frac{s_p^2}{n_x}+\frac{s_p^2}{n_y}}<\mu_X-\mu_Y<(\bar{x}-\bar{y})+t_{n_x+n_y-2,\alpha/2}\sqrt{\frac{s_p^2}{n_x}+\frac{s_p^2}{n_y}}$$

$$(3004-2538)-(2.054)\sqrt{\frac{4427.03}{17}+\frac{4427.03}{14}}<\mu_X-\mu_Y<(3004-2538)+(2.054)\sqrt{\frac{4427.03}{17}+\frac{4427.03}{14}}$$

$$\boxed{416.69<\mu_X-\mu_Y<515.31}$$

We are 95% confident that the mean difference in CPU speed is between 416.69 and 515.31 Mhz.



σ_x^2 and σ_y^2 Unknown, Assumed Unequal

- Population means, independent samples
  - σ_x^2 and σ_y^2 known
  - σ_x^2 and σ_y^2 unknown
    - σ_x^2 and σ_y^2 assumed equal
    - σ_x^2 and σ_y^2 assumed unequal *

Assumptions:

- Samples are randomly and independently drawn
- Populations are normally distributed
- Population variances are unknown and assumed unequal

σ_x^2 and σ_y^2 Unknown, Assumed Unequal

(continued)

- Population means, independent samples
  - σ_x^2 and σ_y^2 known
  - σ_x^2 and σ_y^2 unknown
    - σ_x^2 and σ_y^2 assumed equal
    - σ_x^2 and σ_y^2 assumed unequal *

Forming interval estimates:

- The population variances are assumed unequal, so a pooled variance is not appropriate
- use a t value with ν degrees of freedom, where

$$\nu = \frac{\left[\left(\frac{s_x^2}{n_x}\right) + \left(\frac{s_y^2}{n_y}\right)\right]^2}{\left(\frac{s_x^2}{n_x}\right)^2 / (n_x - 1) + \left(\frac{s_y^2}{n_y}\right)^2 / (n_y - 1)}$$

Confidence Interval, σ_x^2 and σ_y^2 Unknown, Unequal

σ_x^2 and σ_y^2 unknown

σ_x^2 and σ_y^2 assumed equal

σ_x^2 and σ_y^2 assumed unequal *

The confidence interval for $\mu_1 - \mu_2$ is:

$$(\bar{x}-\bar{y}) - t_{\nu,\alpha/2}\sqrt{\frac{s_x^2}{n_x}+\frac{s_y^2}{n_y}} < \mu_X - \mu_Y < (\bar{x}-\bar{y}) + t_{\nu,\alpha/2}\sqrt{\frac{s_x^2}{n_x}+\frac{s_y^2}{n_y}}$$

Where

$$\nu = \frac{\left[\left(\frac{s_x^2}{n_x}\right)+\left(\frac{s_y^2}{n_y}\right)\right]^2}{\left(\frac{s_x^2}{n_x}\right)^2/(n_x-1)+\left(\frac{s_y^2}{n_y}\right)^2/(n_y-1)}$$

Two Population Proportions

Population proportions

Goal: Form a confidence interval for the difference between two population proportions, $\boxed{P_x - P_y}$

Assumptions:

Both sample sizes are large (generally at least 40 observations in each sample)

The point estimate for the difference is $\hat{p}_x - \hat{p}_y$

Two Population Proportions

(continued)

Population proportions

- The random variable

$$Z = \frac{(\hat{p}_x - \hat{p}_y) - (p_x - p_y)}{\sqrt{\frac{\hat{p}_x(1-\hat{p}_x)}{n_x} + \frac{\hat{p}_y(1-\hat{p}_y)}{n_y}}}$$

is approximately normally distributed

Confidence Interval for Two Population Proportions

Population proportions

The confidence limits for $P_x - P_y$ are:

$$(\hat{p}_x - \hat{p}_y) \pm Z_{\alpha/2}\sqrt{\frac{\hat{p}_x(1-\hat{p}_x)}{n_x} + \frac{\hat{p}_y(1-\hat{p}_y)}{n_y}}$$

Example: Two Population Proportions

Form a 90% confidence interval for the difference between the proportion of men and the proportion of women who have college degrees.

- In a random sample, 26 of 50 men and 28 of 40 women had an earned college degree

Example: Two Population Proportions

(continued)

Men: $\hat{p}_x = \dfrac{26}{50} = 0.52$

Women: $\hat{p}_y = \dfrac{28}{40} = 0.70$

$$\sqrt{\frac{\hat{p}_x(1-\hat{p}_x)}{n_x} + \frac{\hat{p}_y(1-\hat{p}_y)}{n_y}} = \sqrt{\frac{0.52(0.48)}{50} + \frac{0.70(0.30)}{40}} = 0.1012$$

For 90% confidence, $Z_{\alpha/2} = 1.645$

Example: Two Population Proportions

(continued)

The confidence limits are:

$$(\hat{p}_x - \hat{p}_y) \pm Z_{\alpha/2}\sqrt{\frac{\hat{p}_x(1-\hat{p}_x)}{n_x} + \frac{\hat{p}_y(1-\hat{p}_y)}{n_y}}$$

$$= (.52 - .70) \pm 1.645(0.1012)$$

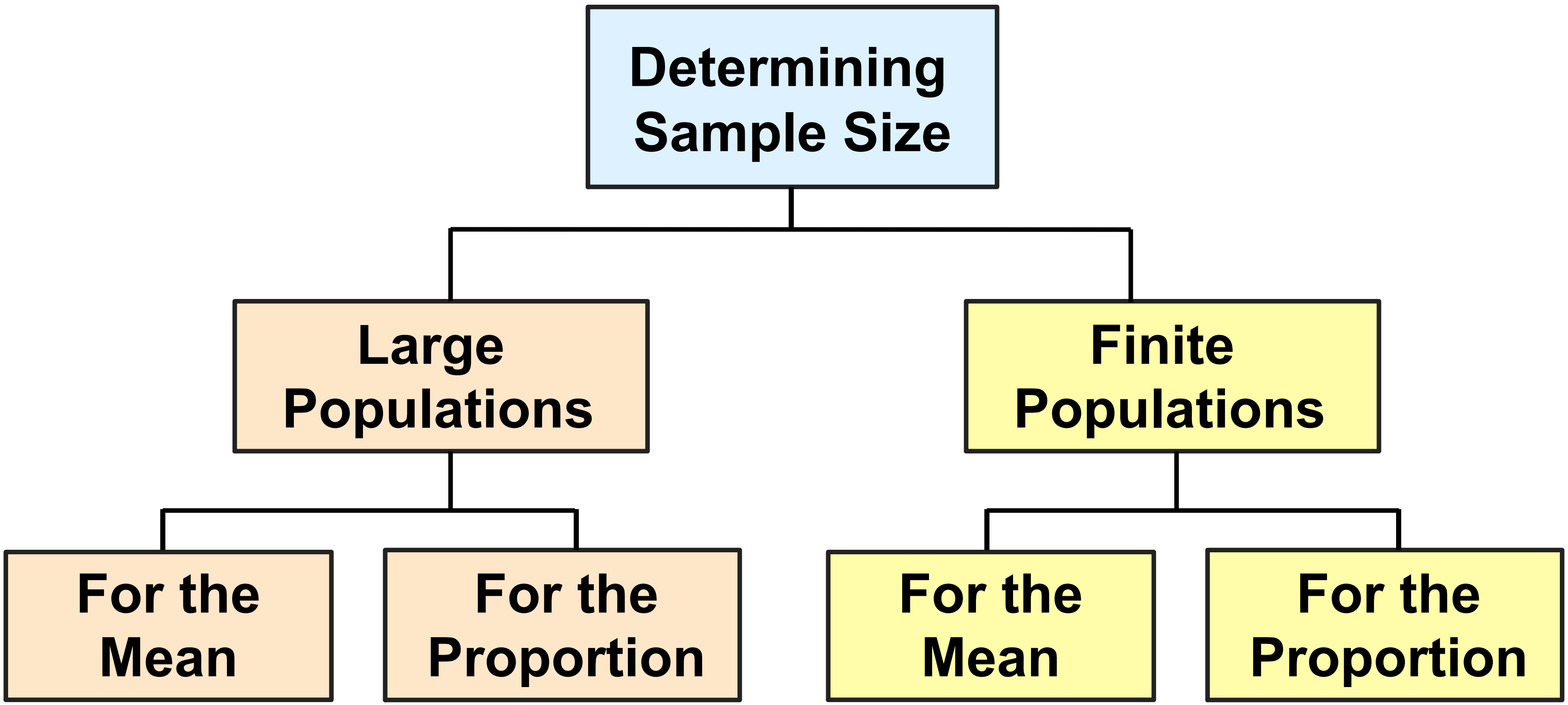
so the confidence interval is

$$-0.3465 < P_x - P_y < -0.0135$$

Since this interval does not contain zero we are 90% confident that the two proportions are not equal

8.4

Sample Size Determination



Margin of Error

- The required sample size can be found to reach a desired margin of error (ME) with a specified level of confidence $(1 - \alpha)$

- The margin of error is also called sampling error
  - the amount of imprecision in the estimate of the population parameter
  - the amount added and subtracted to the point estimate to form the confidence interval

Sample Size Determination

Large Populations

For the Mean

$$\bar{x} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

Margin of Error (sampling error)

$$ME = z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

Sample Size Determination

(continued)

Large Populations

For the Mean

$$ME = z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

Now solve for n to get

$$\boxed{n = \frac{z_{\alpha/2}^2\,\sigma^2}{ME^2}}$$

Sample Size Determination

(continued)

- To determine the required sample size for the mean, you must know:
  - The desired level of confidence ($1 - \alpha$), which determines the $z_{\alpha/2}$ value
  - The acceptable margin of error (sampling error), ME
  - The population standard deviation, σ

Required Sample Size Example

If $\sigma = 45$, what sample size is needed to estimate the mean within ± 5 with 90% confidence?

$$n = \frac{z_{\alpha/2}^2 \, \sigma^2}{ME^2} = \frac{(1.645)^2 (45)^2}{5^2} = 219.19$$

So the required sample size is **n = 220**

(Always round up)

Sample Size Determination: Population Proportion

Large Populations

For the Proportion

$$\hat{p} \pm z_{\alpha/2} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

$$ME = z_{\alpha/2} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

Margin of Error (sampling error)

Sample Size Determination: Population Proportion

(continued)

Large Populations

For the Proportion

$$ME = z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

$\hat{p}(1-\hat{p})$ cannot be larger than 0.25, when $\hat{p} =$ 0.5

Substitute 0.25 for $\hat{p}(1-\hat{p})$ and solve for n to get

$$n = \frac{0.25\, z_{\alpha/2}^2}{ME^2}$$

Sample Size Determination: Population Proportion

(continued)

- The sample and population proportions, $\hat{p}$ and P, are generally not known (since no sample has been taken yet)
- P(1 – P) = 0.25 generates the largest possible margin of error (so guarantees that the resulting sample size will meet the desired level of confidence)
- To determine the required sample size for the proportion, you must know:
  - The desired level of confidence (1 - α), which determines the critical $z_{\alpha/2}$ value
  - The acceptable sampling error (margin of error), ME
  - Estimate P(1 – P) = 0.25

Required Sample Size Example: Population Proportion

How large a sample would be necessary to estimate the true proportion defective in a large population within ±3%, with 95% confidence?

Required Sample Size Example

(continued)

Solution:

For 95% confidence, use $z_{0.025} = 1.96$

$ME = 0.03$

Estimate $P(1 - P) = 0.25$

$$n = \frac{0.25\, z_{\alpha/2}^2}{ME^2} = \frac{(0.25)(1.96)^2}{(0.03)^2} = \boxed{1067.11}$$

So use $n = 1068$

Sample Size Determination: Finite Populations

8.5

Finite Populations

For the Mean

A finite population correction factor is added:

$$\mathrm{Var}(\bar{X}) = \frac{\sigma^2}{n}\left(\frac{N-n}{N-1}\right)$$

1. Calculate the required sample size n_0 using the prior formula:

$$n_0 = \frac{z_{\alpha/2}^2\,\sigma^2}{ME^2}$$

2. Then adjust for the finite population:

$$n = \frac{n_0 N}{n_0 + (N-1)}$$

Sample Size Determination: Finite Populations

Finite Populations

For the Proportion

A finite population correction factor is added:

$$Var(\hat{p}) = \frac{P(1\text{-}P)}{n}\left(\frac{N-n}{N-1}\right)$$

1. Solve for n:

$$n = \frac{NP(1-P)}{(N-1)\sigma_{\hat{p}}^2 + P(1-P)}$$

2. The largest possible value for this expression (if P = 0.25) is:

$$n = \frac{0.25(1-P)}{(N-1)\sigma_{\hat{p}}^2 + 0.25}$$

3. A 95% confidence interval will extend $\pm 1.96\,\sigma_{\hat{p}}$ from the sample proportion

Example: Sample Size to Estimate Population Proportion

(continued)

How large a sample would be necessary to estimate within ±5% the true proportion of college graduates in a population of 850 people with 95% confidence?

Required Sample Size Example

(continued)

Solution:

- For 95% confidence, use $z_{0.025} = 1.96$
- ME = 0.05

$$1.96\,\sigma_{\hat{p}} = 0.05 \quad \Rightarrow \quad \sigma_{\hat{p}} = 0.02551$$

$$n_{max} = \frac{0.25N}{(N-1)\sigma_{\hat{p}}^2 + 0.25} = \frac{(0.25)(850)}{(849)(0.02551)^2 + 0.25} = 264.8$$

So use n = 265

Chapter Summary

- Compared two dependent samples (paired samples)
  - Formed confidence intervals for the paired difference
- Compared two independent samples
  - Formed confidence intervals for the difference between two means, population variance known, using z
  - Formed confidence intervals for the differences between two means, population variance unknown, using t
  - Formed confidence intervals for the differences between two population proportions
- Formed confidence intervals for the population variance using the chi-square distribution
- Determined required sample size to meet confidence and margin of error requirements