Part II

Portfolio Theory

2.(a) Risk and Return

Financial Markets and Investments

Mercados e Investimentos Financeiros

Adapted from Bodie, Kane and Marcus' slides, McGraw-Hill/Irwin

1. Risk and Return

1.1 Determinants of the Level of Interest Rates

- Factors Influencing Rates
  - Supply of funds
    - Households
  - Demand for funds
    - Businesses
  - Government's Net Supply and/or Demand
    - Central Bank Actions

1. Risk and Return

1.1 Determinants of the Level of Interest Rates

- Real and Nominal Rates of Interest
  - Nominal interest rate
    - Growth rate of your money
  - Real interest rate
    - Growth rate of your purchasing power
  - If *R* is the nominal rate and *r* the real rate and *i* is the inflation rate (?):

$$r \approx R - i$$

  - However, the exact relationship is

$$1 + r = \frac{1 + R}{1 + i} \Leftrightarrow r = \frac{R - i}{1 + i}$$

1. Risk and Return

1.1 Determinants of the Level of Interest Rates

- Equilibrium Real Rate of Interest
  - Determined by:
    - Supply
    - Demand
    - Government actions
    - Expected rate of inflation

1. Risk and Return

1.1 Determinants of the Level of Interest Rates

- Determination of the Equilibrium Real Rate of Interest

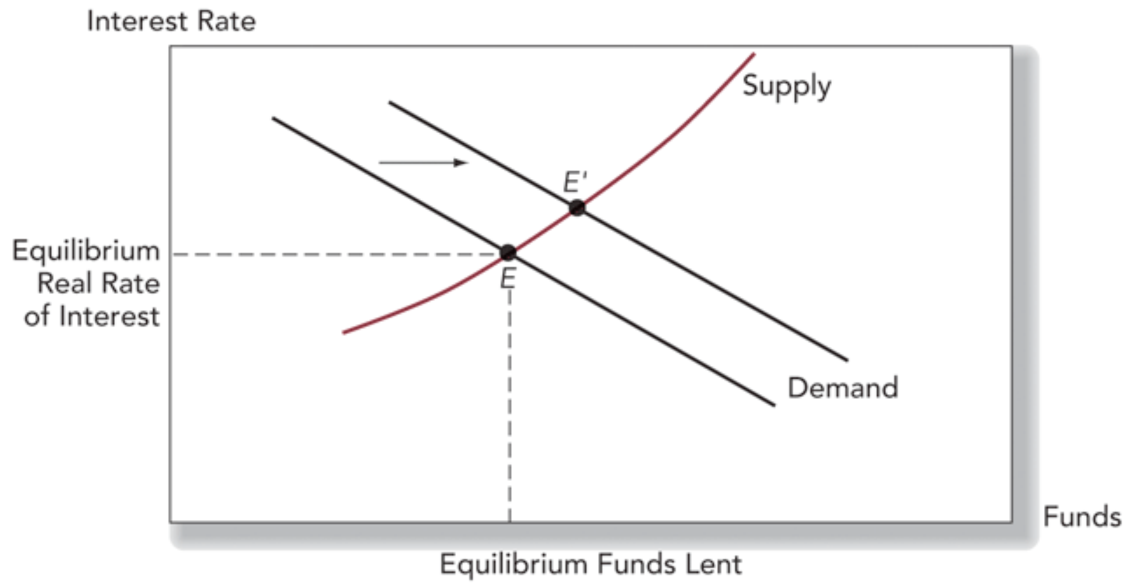

FIGURE 5.1 Determination of the equilibrium real rate of interest

1.1 Determinants of the Level of Interest Rates

- Equilibrium Nominal Rate of Interest
  - As the inflation rate increases, investors will demand higher nominal rates of return
  - If *E*(*i*) denotes current expectations of inflation, then we get the Fisher Equation:

$$r = R - E(i)$$

1. Risk and Return

1.1 Determinants of the Level of Interest Rates

- Taxes and the Real Rate of Interest
  - Tax liabilities are based on nominal income
    - Given a tax rate (t), nominal interest rate (R), after-tax interest rate is $R(1\text{-}t)$
    - Real after-tax rate is:

$$R(1 - t) - i = (r + i)(1 - t) - i = r(1 - t) - it$$

1.2 Comparing Rates of Return for Different Holding Periods

- Risk Free Zero Coupon Bond Rate of Return, with a maturity of **T years**

$$r_f(T) = \frac{FV}{P(T)} - 1$$

1. Risk and Return

1.2 Comparing Rates of Return for Different Holding Periods

- Effective Annual Rate (EAR), *T<1*

$$EAR = \left[1 + r_f(T)\right]^{\frac{1}{T}} - 1$$

- Annual Percentage Rates (APR)

$$APR = \frac{(1 + EAR)^T - 1}{T}$$

- Continuous Compounding

$$1 + EAR = e^{r_{cc}} \wedge r_{cc} = \ln(1 + EAR)$$

1. Risk and Return

1.2 Comparing Rates of Return for Different Holding Periods

Compounding Period	T	EAR = $[1 + r_f(T)]^{1/T} - 1 = .058$		APR = $r_f(T)*(1/T) = .058$	
		$r_f(T)$	APR = $[(1 + \text{EAR})^\wedge T - 1]/T$	$r_f(T)$	EAR = $(1 + \text{APR}*T)^\wedge(1/T) - 1$
1 year	1.0000	.0580	.05800	.0580	.05800
6 months	0.5000	.0286	.05718	.0290	.05884
1 quarter	0.2500	.0142	.05678	.0145	.05927
1 month	0.0833	.0047	.05651	.0048	.05957
1 week	0.0192	.0011	.05641	.0011	.05968
1 day	0.0027	.0002	.05638	.0002	.05971
Continuous			$r_{cc} = \ln(1 + \text{EAR}) = .05638$		EAR = $\exp(r_{cc}) - 1 = .05971$

TABLE 5.1

Annual percentage rate (APR) and effective annual rates (EAR). In the first set of columns, we hold the equivalent annual rate (EAR) fixed at 5.8%, and find APR for each holding period. In the second set of columns, we hold APR fixed and solve for EAR.

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1.2 Comparing Rates of Return for Different Holding Periods

Effective Annual Rate with partial payments during the year

Consider a financial investment whose annual return is 10% and that every semester pays interest, that is reinvested. What is the **effective annual rate**?

end 1st semester: $100(1+0.05)=105$

end of the year: $105(1+0.05)=105(1+0.05)^2=110.25$

Effective Annual Rate= (110.25-100)/100=0.1025

→ **10.25%**

1. Risk and Return

1.2 Comparing Rates of Return for Different Holding Periods

- Assume the annual return is given (*APR* = annual percentage rate) but compounding happens *m* times per year. What is the Effective Annual Rate (EAR)?

$$\left(1+\frac{APR}{m}\right)^{m}=1+EAR$$

Number payments per year (**m**)	EAR
1	10%
2	10.25%
3	10.34%
4	10.38%
12	10.47%
Continuously compounding	10.52%

- In the limit as m $\rightarrow\infty$, EAR = e^r-1

1. Risk and Return

1.3 Bills and Inflation, 1926-2005

- Entire post-1926 history of annual rates:
  - www.mhhe.com/bkm
- Average real rate of return on T-bills for the entire period was 0.72 percent
- Real rates are larger in late periods

1.3 Bills and Inflation, 1926-2005

Portfolio	Statistic	1926–2005	1966–2005	1981–2005	1971–1995	1961–1985	1951–1975	1941–1965	1931–1955	1926–1950
U.S. T-bills	Average	3.75	5.98	5.73	7.04	6.55	3.66	1.62	0.63	1.02
	Standard Deviation	3.15	2.84	3.15	2.87	3.15	1.97	1.16	0.57	1.33
	Serial Correlation	0.91	0.81	0.87	0.76	0.82	0.82	0.85	0.85	0.88
U.S. CPI inflation	Average	3.13	4.70	3.36	5.60	5.39	3.28	3.39	2.21	1.51
	Standard Deviation	4.29	3.02	1.62	3.38	3.63	2.99	4.28	5.75	6.02
	Serial Correlation	0.64	0.73	0.32	0.68	0.74	0.69	0.35	0.46	0.52
U.S. real rate	Average	0.72	1.25	2.28	1.41	1.14	0.40	−1.54	−1.25	−0.13
	Standard Deviation	3.97	2.35	2.18	2.76	2.60	1.61	4.40	5.68	6.35
	Serial Correlation	0.66	0.71	0.72	0.71	0.76	0.49	0.53	0.51	0.64

TABLE 5.2

History of T-bill rates, inflation, and real rates for generations, 1926–2005
Sources: *T-bills:* Salomon Smith Barney 3-month U.S. T-bill index; *inflation data:* Bureau of Labor Statistics.

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1. Risk and Return

1.3 Bills and Inflation, 1926-2005

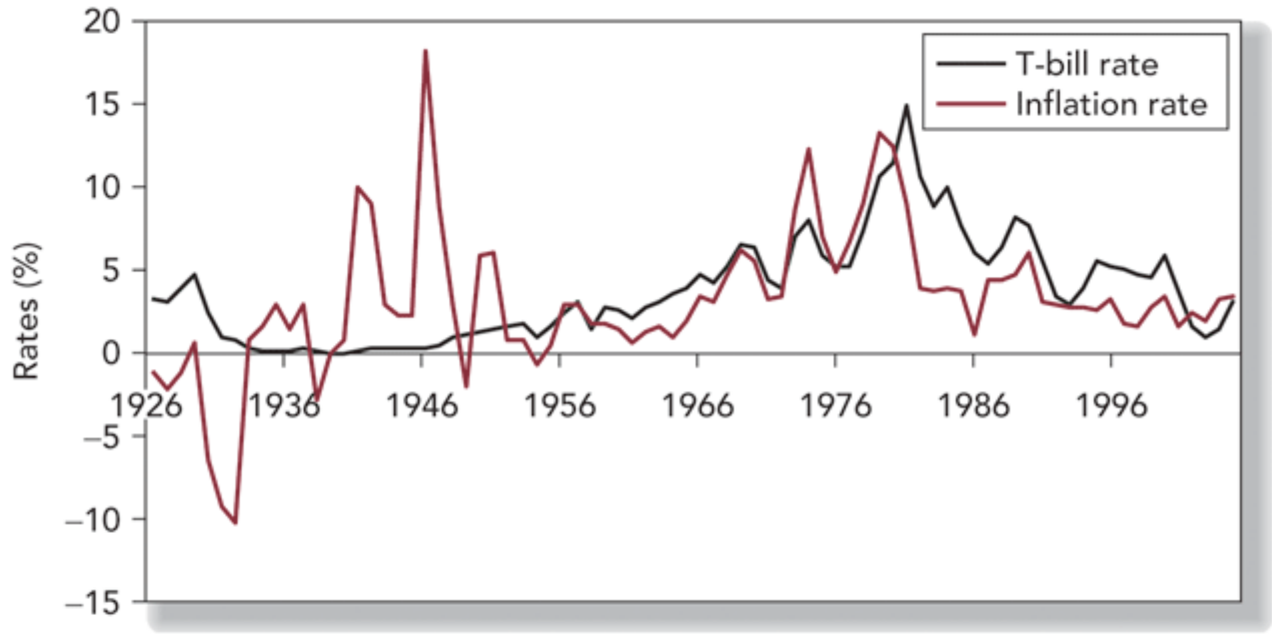


FIGURE 5.2 Interest and inflation rates, 1926–2005

1. Risk and Return

1.3 Bills and Inflation, 1926-2005

Wealth index: $\$1(1+r)^t$

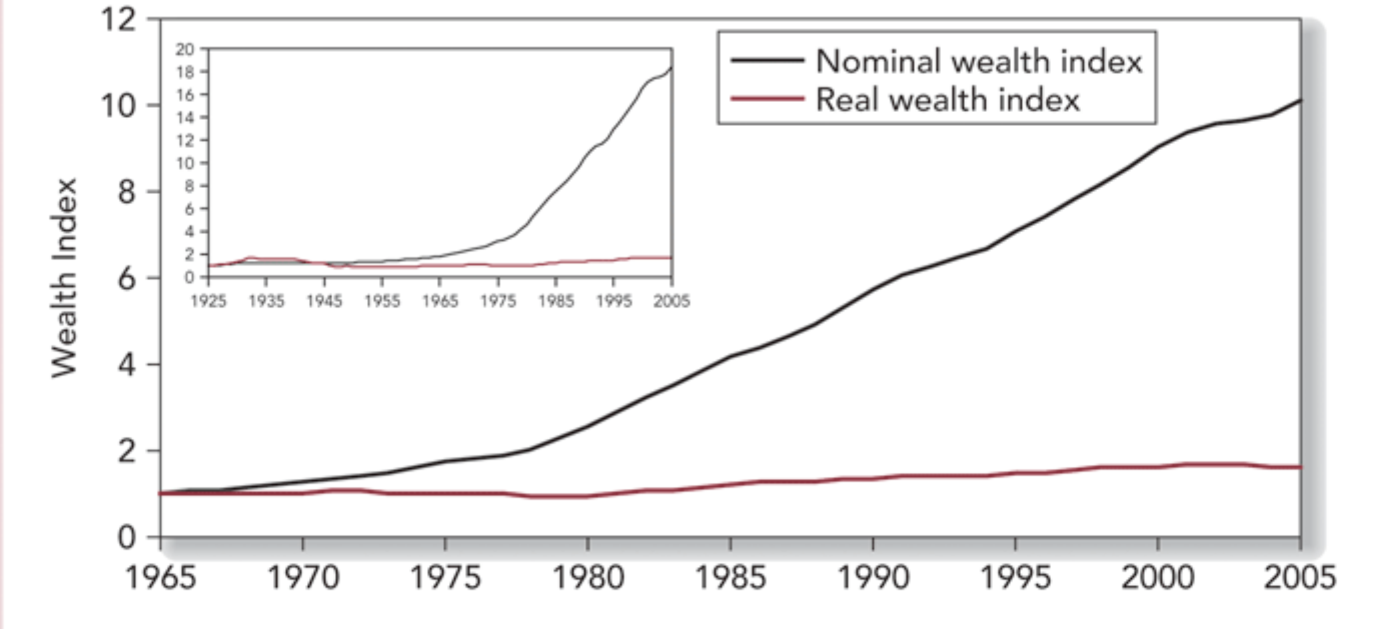


FIGURE 5.3 Nominal and real wealth indexes for investments in Treasury bills, 1966–2005 (inset figure is for 1925–2005)

1.4 Risk and Risk Premiums

- Rates of Return: Single Period

$$HPR = \frac{P_1 - P_0 + D_1}{P_0}$$

HPR = Holding Period Return

P_0 = Beginning price

P_1 = Ending price

D_1 = Dividend during period one

1.4 Risk and Risk Premiums

- Expected Return

$$E(r) = \sum_{s} p(s) r(s)$$

- Variance, $E[(r(s)-E(r))^2]$

$$\sigma^2 = \sum_{s} p(s)\left[r(s) - E(r)\right]^2$$

- Standard Deviation

$$\sigma = \sqrt{\sigma^2} = \sqrt{\sum_{s} p(s)\left[r(s) - E(r)\right]^2}$$

$p(s)$ = probability of a state
$r(s)$ = return if a state occurs
s = state

1.4 Risk and Risk Premiums

State	Prob. of State	*r* in State
1	.1	-.05
2	.2	.05
3	.4	.15
4	.2	.25
5	.1	.35

$E(r) = (.1)(-.05) + (.2)(.05) \ldots + (.1)(.35)$

$E(r) = .15$

$\text{Var} = [(.1)(-.05-.15)^2 + (.2)(.05-.15)^2 \ldots + .1(.35-.15)^2] = .01199$

$\text{S.D.} = [.01199]^{1/2} = .1095$

1. Risk and Return

1.4 Risk and Risk Premiums

r_f: Risk free rate; *HPR*: Hold period return

- Risk Premium (*RP*)

$$RP = E\left(HPR\right) - r_f$$

- Excess Return (*ER*)

$$ER = HPR - r_f$$

1. Risk and Return

1.5 Time Series Analysis of Past Rates of Return

- The Arithmetic Average Rate of Return $(\bar{r})$

$$\bar{r} = \frac{1}{n}\sum_{t=1}^{n} r(t)$$

- The Geometric Average Rate of Return (**g**)

$$TV_n = (1+r_1)(1+r_2)\times\ldots\times(1+r_n)$$ ← Terminal Value of the Investment

$$g = TV^{1/n} - 1$$

1. Risk and Return

1.5 Time Series Analysis of Past Rates of Return

- Variance = expected value of squared deviations

$$\sigma^2 = \frac{1}{n}\sum_{s=1}^{n}[r(s) - \bar{r}]^2$$

- When eliminating the bias, Variance and Standard Deviation become:

$$\sigma = \sqrt{\frac{1}{n-1}\sum_{j=1}^{n}[r(s) - \bar{r}]^2}$$

1. Risk and Return

1.5 Time Series Analysis of Past Rates of Return

- The Reward-to-Volatility (Sharpe) Ratio

$$\text{Sharpe Ratio for Portfolios} = \frac{\text{Risk Premium}}{\text{SD of Excess Return}}$$

1.6 Estimating Expect Returns from Data

Example: Initial investment=$100; rate of return on the 1st year=10%; rate of return on the 2nd year=-10%. What is the average rate of return?

V_1=P0(1+0.1)=110

V_2=P1(1-0.1)=110(0.9)=99

Return over the two periods:

$$R = \frac{99 - 100}{100} = -1\%$$

1. Risk and Return

1.6 Estimating Expect Returns from Data

- The Arithmetic Average

$$\bar{r} = \frac{0{,}1 - 0{,}1}{2} = 0$$

- The Geometric Average

$$g = \sqrt{1{,}1 \times 0{,}9} - 1 = -0{,}005$$

- Arithmetic Average ≥ Geometric Average

- The Geometric Average is the appropriate measure in order to evaluate past returns

- But, which measure should be used to estimate expected returns?

1.6 Estimating Expect Returns from Data

- According to Jacquier, Kane and Marcus "Geometric or Arithmetic Mean: A Reconsideration", *Financial Analysts Journal*, Nov 2003, the non-biased estimator of the return is:

Arithmetic Mean × (1-H/T)+ Geometric Mean × (H/T),

T: time horizon of the historical data

H: time horizon of the forecast

1. Risk and Return

1.7 Normal Distribution

- (If) Rate of return is distributed according to a normal distribution, *i.e.*, $R \sim N(\mu, \sigma^2)$.

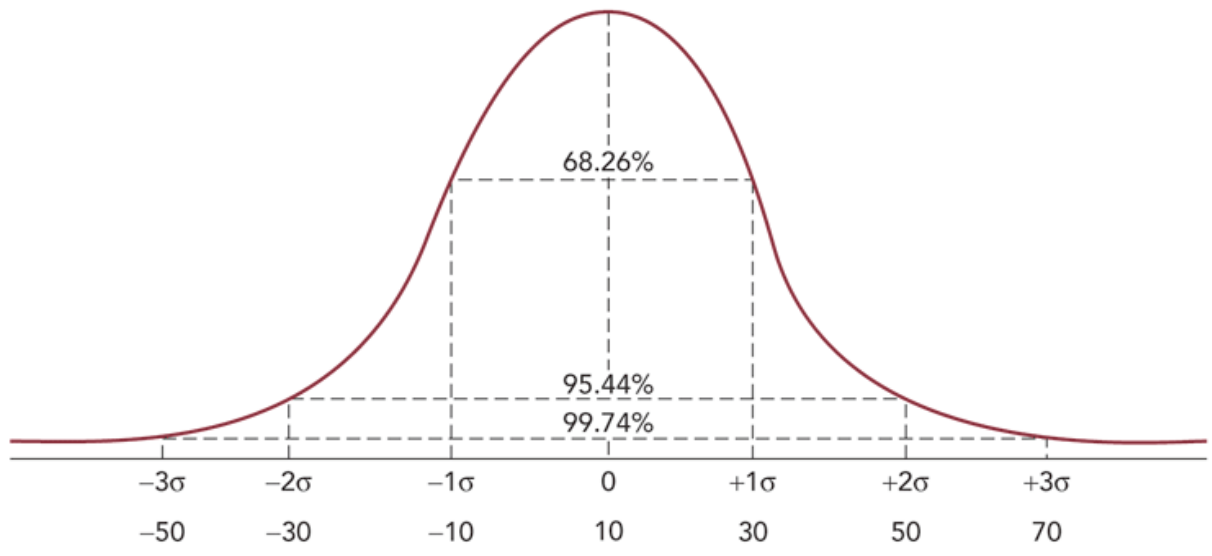


FIGURE 5.4 The normal distribution

1.7 Normal Distribution

- Deviations from normality….

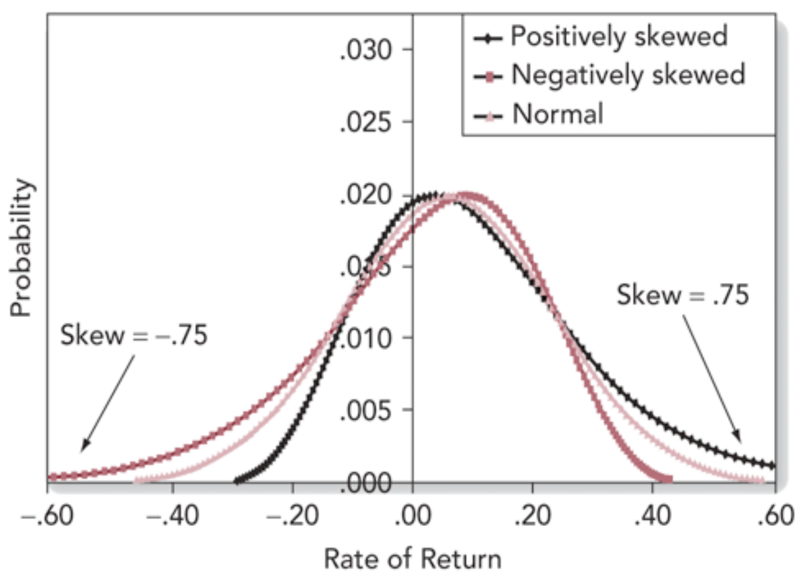


FIGURE 5.5A Normal and skewed distributions (mean = 6%, SD = 17%)

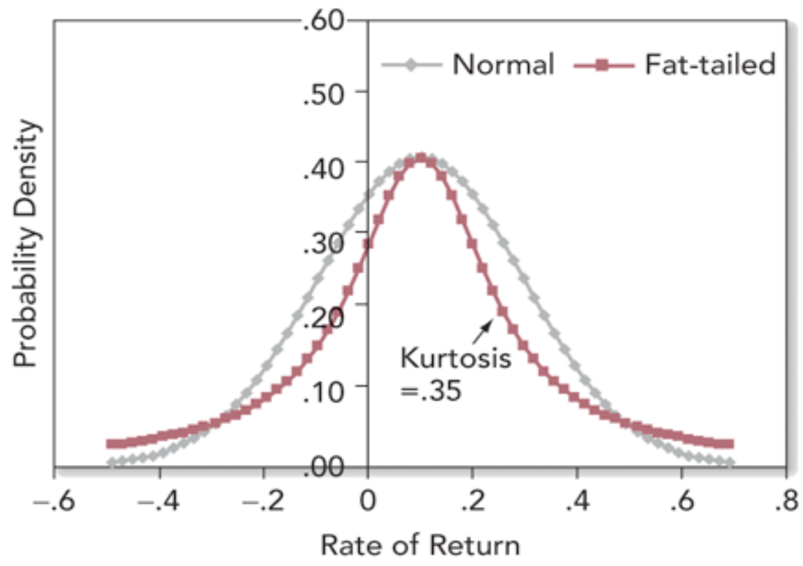


FIGURE 5.5B Normal and fat-tailed distributions (mean = .1, SD = .2)

1. Risk and Return

1.7 Normal Distribution

- Problem with the normality assumption…

$$r \sim N \Rightarrow P(R < -1) > 0$$

- Log-normality solves this problem!
- Returns follow a log-normal distribution ($X \sim N \Rightarrow e^X \sim LN$)

$$1 + r \sim \ln N \Leftrightarrow \ln(1 + r) \sim N$$

1. Risk and Return

1.8 Historical Records

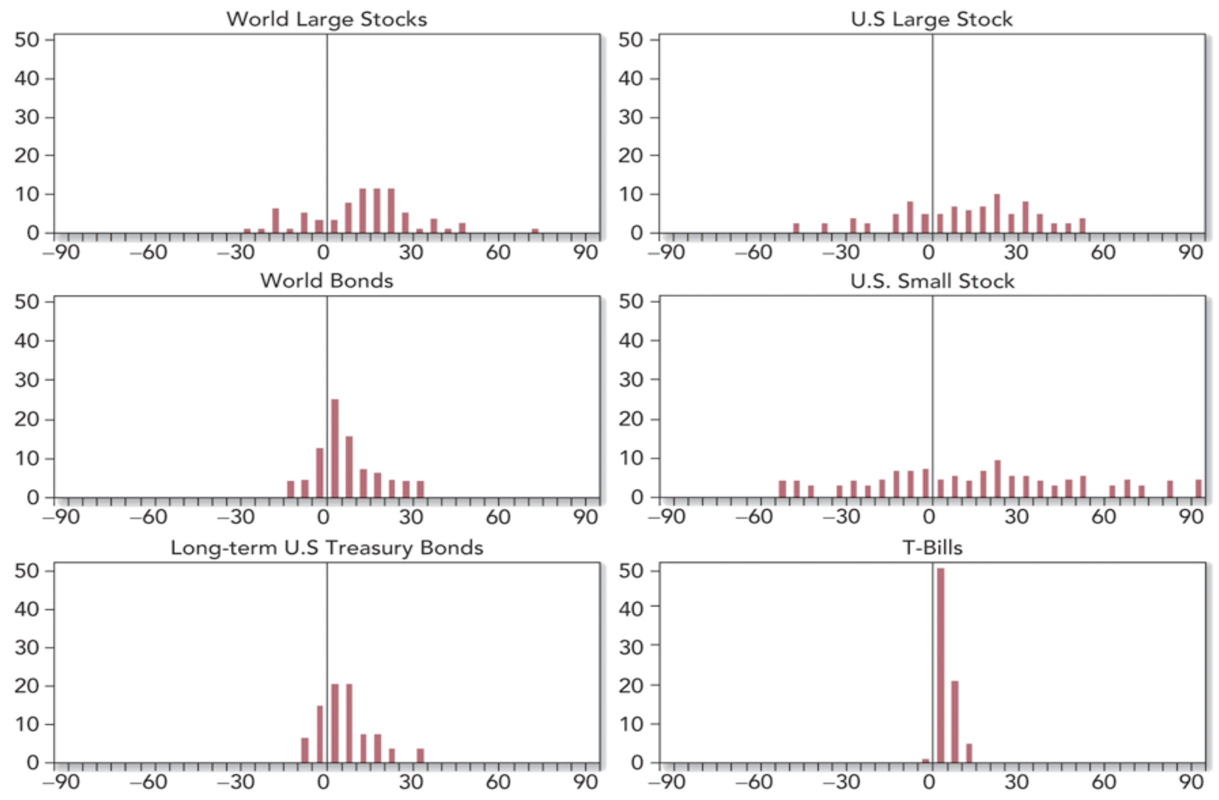


FIGURE 5.6 Frequency distributions of rates of return for 1926–2005

1. Risk and Return

1.8 Historical Records

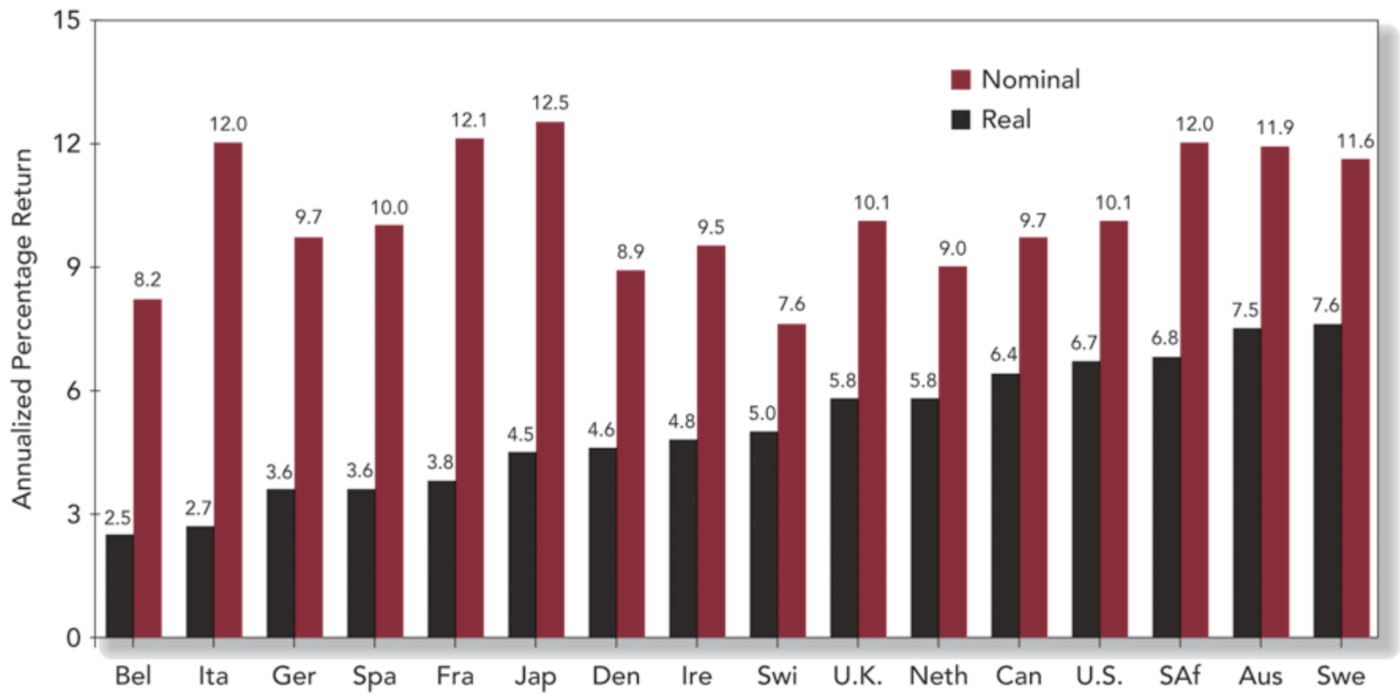


FIGURE 5.7 Nominal and real equity returns around the world, 1900–2000

Source: Elroy Dimson, Paul Marsh, and Mike Staunton, *Triumph of the Optimists: 101 Years of Global Investment Returns* (Princeton University Press, 2002), p. 50. Reprinted by permission of the Princeton University Press.

1. Risk and Return

1.8 Historical Records

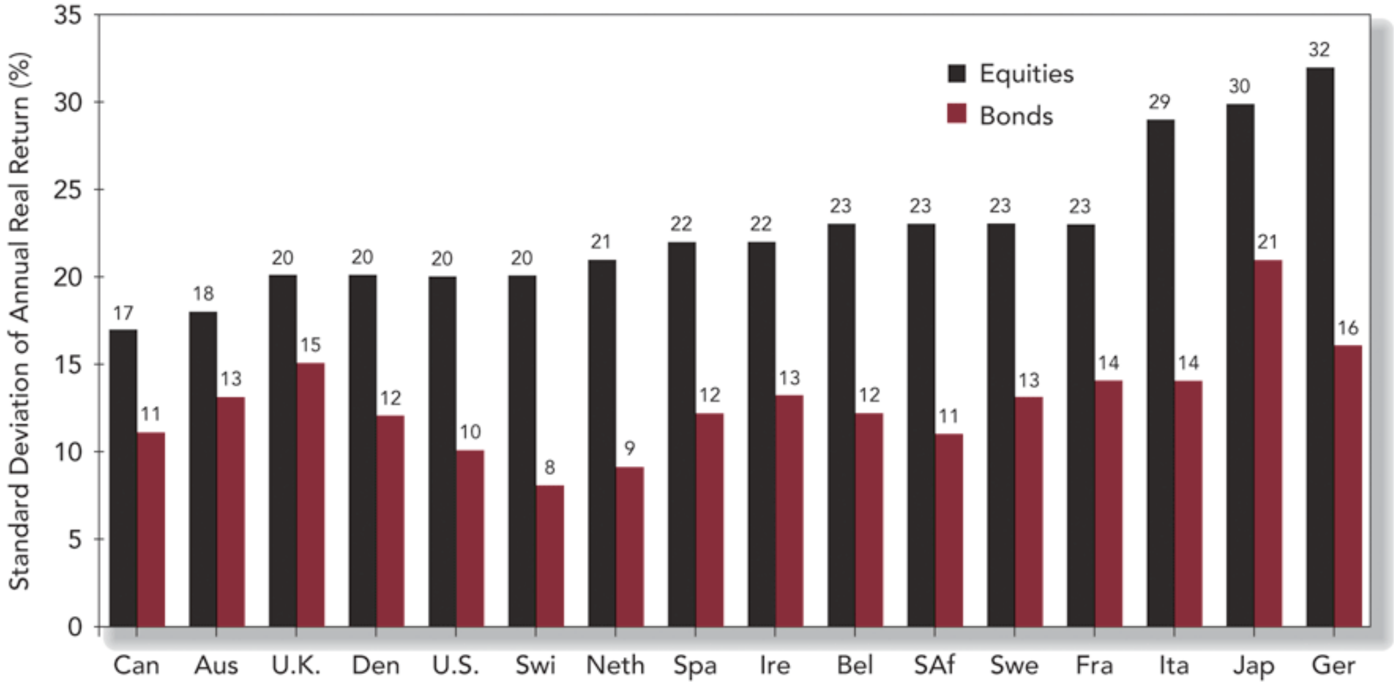


FIGURE 5.8 Standard deviations of real equity and bond returns around the world, 1900–2000

Source: Elroy Dimson, Paul Marsh, and Mike Staunton, *Triumph of the Optimists: 101 Years of Global Investment Returns* (Princeton University Press, 2002), p. 61. Reprinted by permission of the Princeton University Press.

1.8 Historical Records

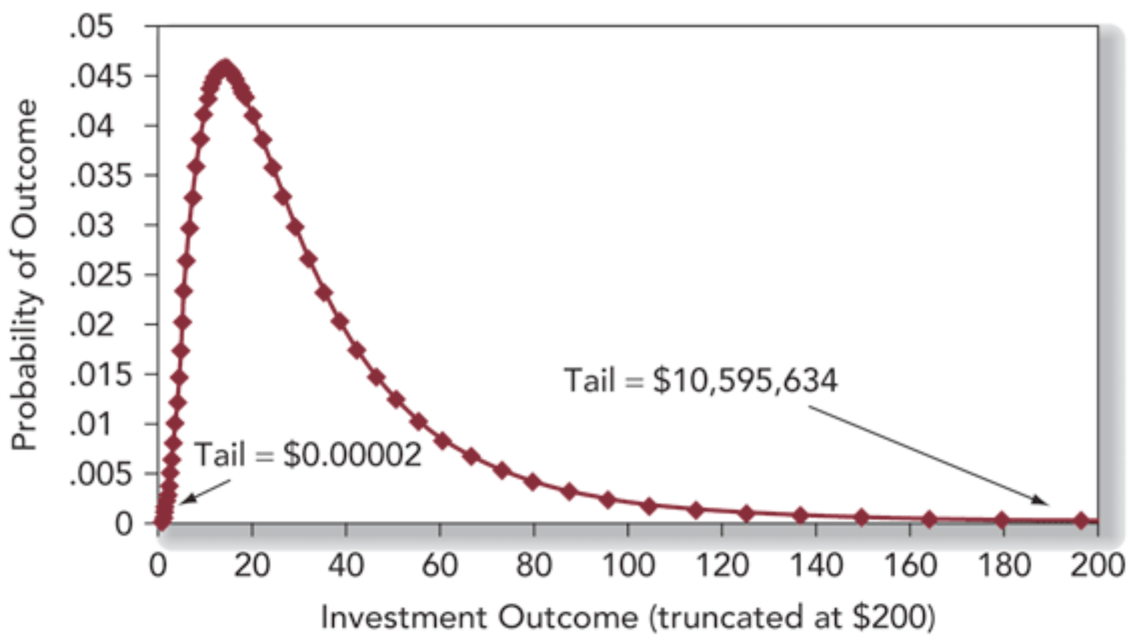
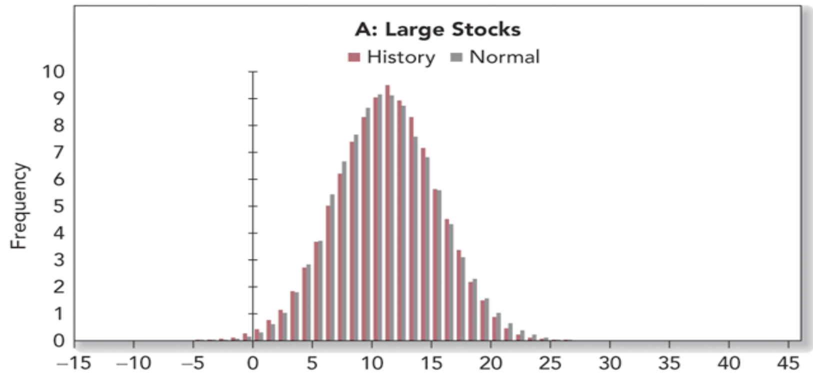


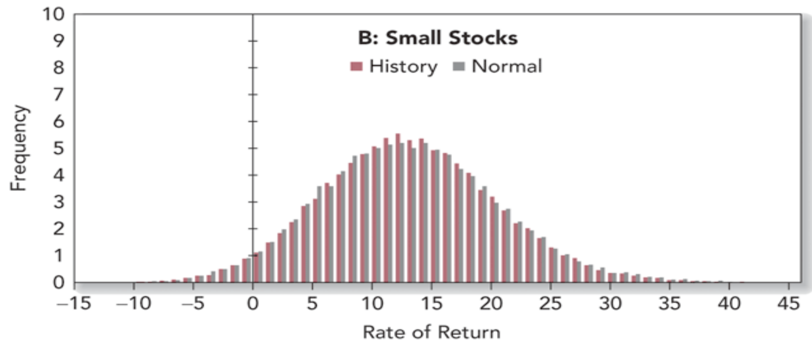
FIGURE 5.9 Probability of investment outcomes after 25 years with a lognormal distribution (approximated from a binomial tree)

1.8 Historical Records



	Actual*	Normal
Avg. 1-year HPR	12.13	12.15
Avg. 25-year HPR	10.24	10.29
SD	4.27	4.32
Min	–11.53	–6.67
Terminal loss (%)	95	82
Max	28.88	29.32
Probability (loss)	.0095	.0064
Probability (shortfall)	.1044	.0603
Skew	–.0854	.1135
Kurtosis	.0040	–.0121

*Bootstrapped history



	Actual*	Normal
Avg. 1-year HPR	17.97	17.95
Avg. 25-year HPR	12.28	12.21
SD	7.28	7.41
Min	–17	–14.9
Terminal loss (%)	99	98
Max	47.03	48.34
Probability (loss)	.0415	.0428
Probability (shortfall)	.1178	.1232
Skew	.1362	.2000
Kurtosis	.0678	.0598

*Bootstrapped history

FIGURE 5.10 Annually compounded, 25-year HPRs from bootstrapped history and a normal distribution (50,000 observations)

1. Risk and Return

1.8 Historical Records

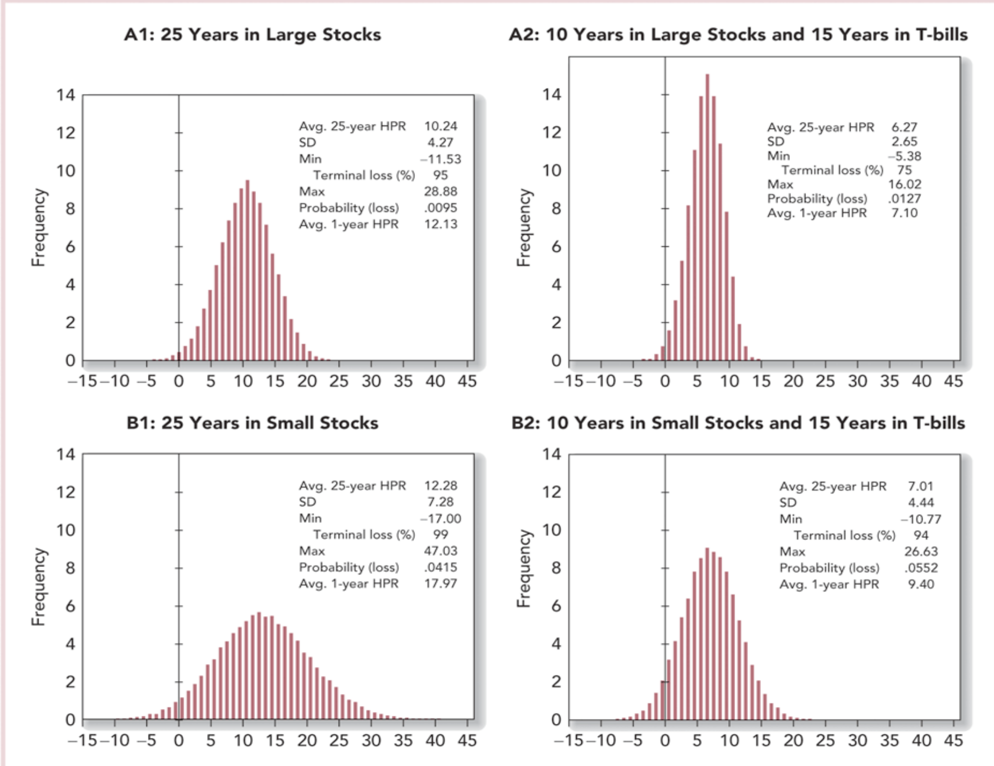


FIGURE 5.11 Annually compounded, 25-year HPRs from bootstrapped history (50,000 observations)