Part II

Portfolio Theory

2c. Optimal Risky Portfolios

Financial Markets and Investments

Adapted from Bodie, Kane and Marcus' slides, McGraw-Hill/Irwin

3. Optimal Risky Portfolios

3.1. Diversification and Portfolio Risk

- Market risk
  - Systematic or nondiversifiable, not eliminated
- Firm-specific risk
  - Diversifiable or nonsystematic, can be reduced (e.g. with the number of stocks)

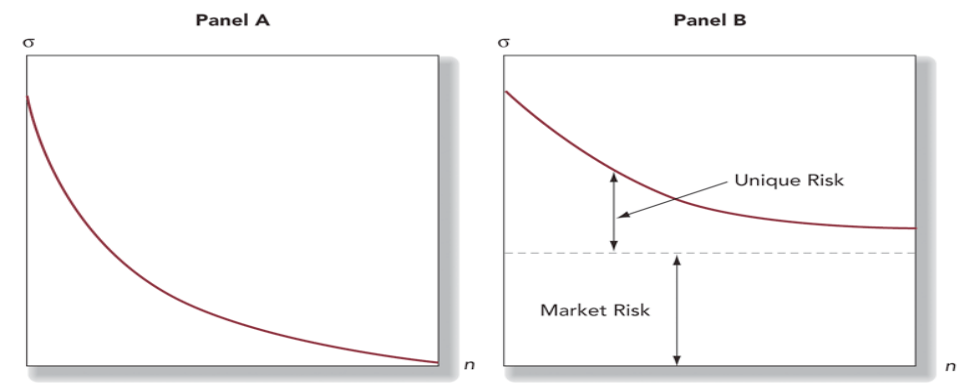

FIGURE 7.1 Portfolio risk as a function of the number of stocks in the portfolio

3. Optimal Risky Portfolios

3.1. Diversification and Portfolio Risk

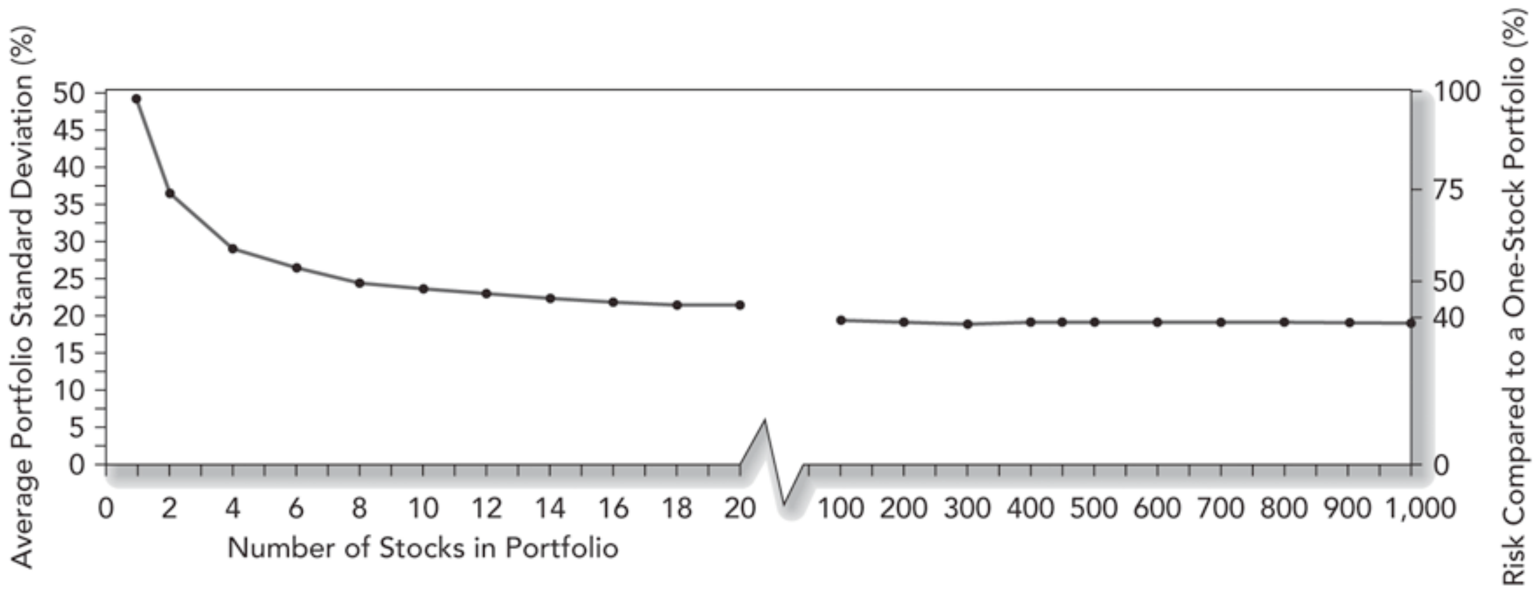

FIGURE 7.2 Portfolio diversification. The average standard deviation of returns of portfolios composed of only one stock was 49.2%. The average portfolio risk fell rapidly as the number of stocks included in the portfolio increased. In the limit, portfolio risk could be reduced to only 19.2%.

Source: From Meir Statman, "How Many Stocks Make a Diversified Portfolio? *Journal of Financial and Quantitative Analysis* 22 (September 1987). Reprinted by permission.

3. Optimal Risky Portfolios

3.2. Efficient Portfolios: Two Risky Assets

- Example, Bond & Stock mutual funds

	Debt	Equity
Expected return, $E(r)$	8%	13%
Standard deviation, σ	12%	20%
Covariance, $Cov(r_D, r_E)$	72	
Correlation coefficient, ρ_{DE}	.30	

TABLE 7.1

Descriptive statistics for two mutual funds

3. Optimal Risky Portfolios

3.2. Efficient Portfolios: Two Risky Assets

- Return composition: $r_P = w_D\, r_D + w_E r_E$; $(w_{E1} - w_D)$
- Expected Return

$$E(r_p) = w_D E(r_D) + w_E E(r_E)$$

r_P = Portfolio Return
w_D = Bond Weight
r_D = Bond Return
w_E = Equity Weight
r_E = Equity Return

Example (naïve portfolio)

$$E(r_p) = w_D E(r_D) + w_E E(r_E)$$
$$= 0{,}5 \times 8 + 0{,}5 \times 13$$
$$= 10{,}5$$

3. Optimal Risky Portfolios

3.2. Efficient Portfolios: Two Risky Assets

- Total Risk
  - Volatility, measured by standard deviation

$$\sigma_P^2 = w_D^2\sigma_D^2 + w_E^2\sigma_E^2 + 2w_D w_E Cov(r_D, r_E)$$

σ_D^2 = Variance of Security D

σ_E^2 = Variance of Security E

$Cov(r_D, r_E)$ = Covariance of returns for Security D and Security E

Example (naïve portfolio)

$$\sigma_P^2 = w_D^2\sigma_D^2 + w_E^2\sigma_E^2 + 2w_D w_E Cov(r_D, r_E)$$

$$= 0{,}5^2 \times 12^2 + 0{,}5^2 \times 20^2 + 2 \times 0.5 \times 0.5 \times 72$$

$$= 172$$

$$\sigma_p = \sqrt{172} = 13.11$$

3.2. Efficient Portfolios: Two Risky Assets

- Correlation Coefficient

$$\rho_{DE} = \frac{\sigma_{DE}}{\sigma_D \sigma_E}, \qquad \rho \in [-1,1]$$

- If ρ_{DE} = 1.0, the securities would be perfectly positively correlated
- If ρ_{DE} = - 1.0, the securities would be perfectly negatively correlated

3.2. Efficient Portfolios: Two Risky Assets

- Covariance and Correlation
  - Portfolio risk depends on the correlation between the returns of the assets in the portfolio
  - Covariance and the correlation coefficient provide a measure of the way returns two assets vary
  - $\sigma_P^2 = w_D^2\sigma_D^2 + w_E^2\sigma_E^2 + 2w_D w_E \sigma_D \sigma_E \rho_{DE}$
  - $\sigma_P^2 = (w_D\sigma_D + w_E\sigma_E)^2$ if ρ_{DE}=1
  - $\sigma_P^2 = (w_D\sigma_D - w_E\sigma_E)^2$ if ρ_{DE}=-1

3. Optimal Risky Portfolios

3.2. Efficient Portfolios: Two Risky Assets

TABLE 7.3

Expected return and standard deviation with various correlation coefficients

w_D	w_E	$E(r_P)$	Portfolio Standard Deviation for Given Correlation			
			$\rho = -1$	$\rho = 0$	$\rho = .30$	$\rho = 1$
0.00	1.00	13.00	20.00	20.00	20.00	20.00
0.10	0.90	12.50	16.80	18.04	18.40	19.20
0.20	0.80	12.00	13.60	16.18	16.88	18.40
0.30	0.70	11.50	10.40	14.46	15.47	17.60
0.40	0.60	11.00	7.20	12.92	14.20	16.80
0.50	0.50	10.50	4.00	11.66	13.11	16.00
0.60	0.40	10.00	0.80	10.76	12.26	15.20
0.70	0.30	9.50	2.40	10.32	11.70	14.40
0.80	0.20	9.00	5.60	10.40	11.45	13.60
0.90	0.10	8.50	8.80	10.98	11.56	12.80
1.00	0.00	8.00	12.00	12.00	12.00	12.00
			Minimum Variance Portfolio			
		w_D	0.6250	0.7353	0.8200	—
		w_E	0.3750	0.2647	0.1800	—
		$E(r_P)$	9.8750	9.3235	8.9000	—
		σ_P	0.0000	10.2899	11.4473	—

3. Optimal Risky Portfolios

3.2. Efficient Portfolios: Two Risky Assets; Expected Return effect

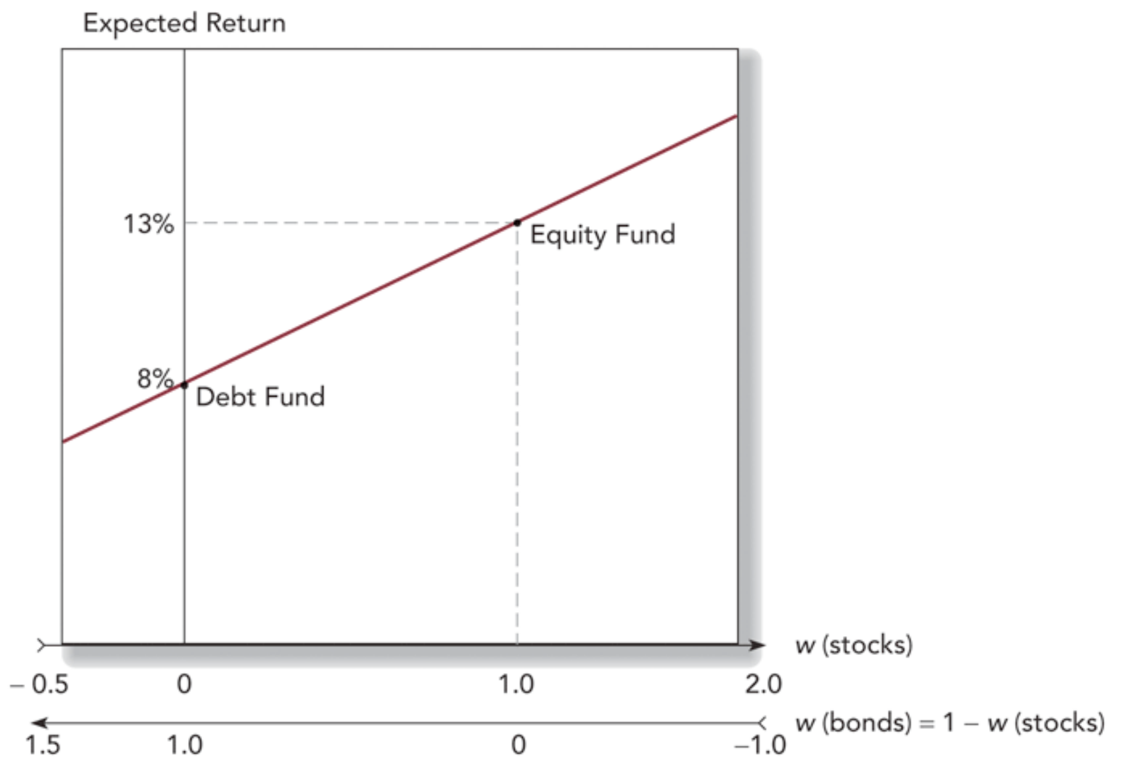

FIGURE 7.3 Portfolio expected return as a function of investment proportions

3.2. Efficient Portfolios: Two Risky Assets; SD effect

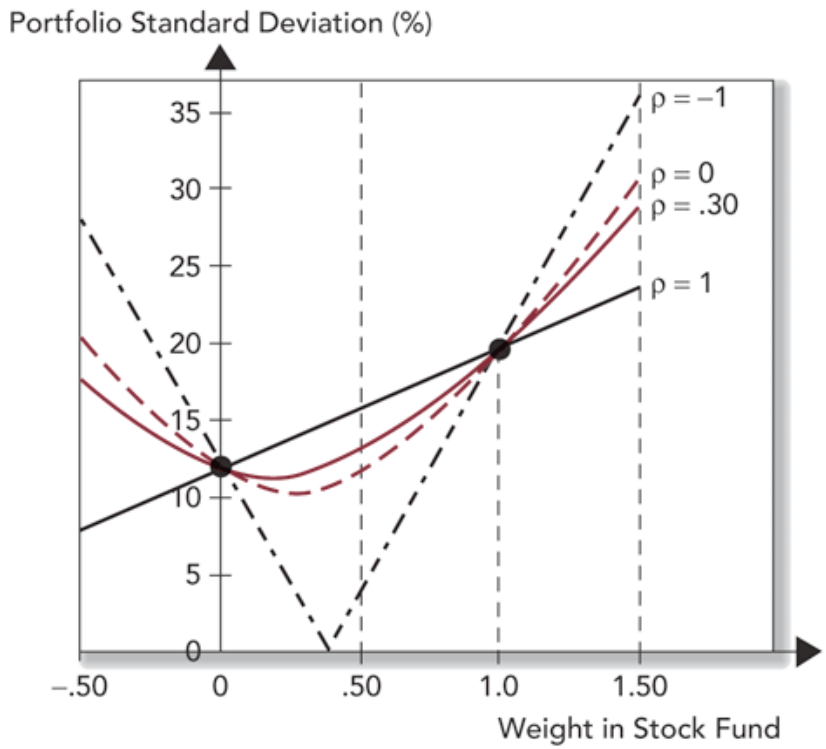

FIGURE 7.4 Portfolio standard deviation as a function of investment proportions

3. Optimal Risky Portfolios

3.2. Efficient Portfolios: Two Risky Assets; Combine ER & SD

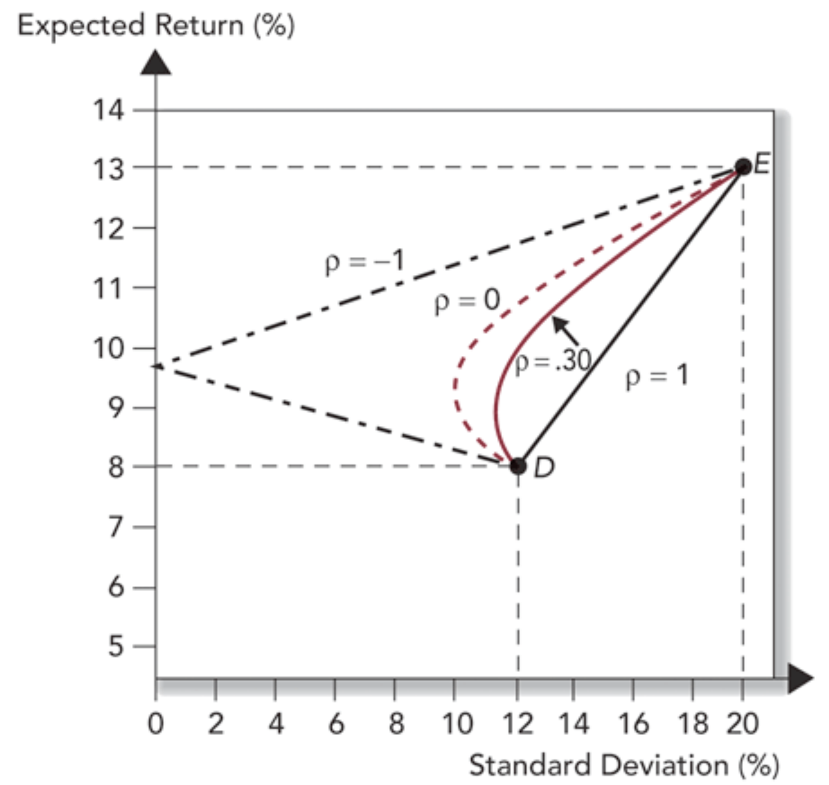

FIGURE 7.5 Portfolio expected return as a function of standard deviation

3.2. Efficient Portfolios: Two Risky Assets

- Correlation Effects
  - The relationship depends on the correlation coefficient ($-1.0 \leq \rho \leq +1.0$)
  - The smaller the correlation, the greater the risk reduction potential
  - If $\rho = +1.0$, no risk reduction is possible

3.2. Efficient Portfolios: Two Risky Assets

- Two Risky Assets and a Risk Free Asset

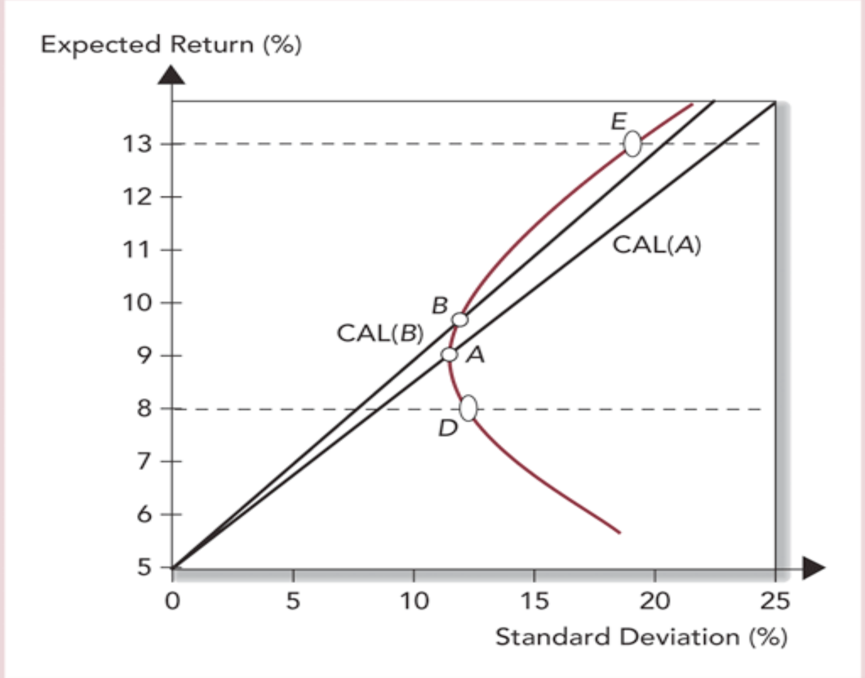

FIGURE 7.6 The opportunity set of the debt and equity funds and two feasible CALs

3. Optimal Risky Portfolios

3.2. Efficient Portfolios: Two Risky Assets

- Two Risky Assets and a Risk Free Asset

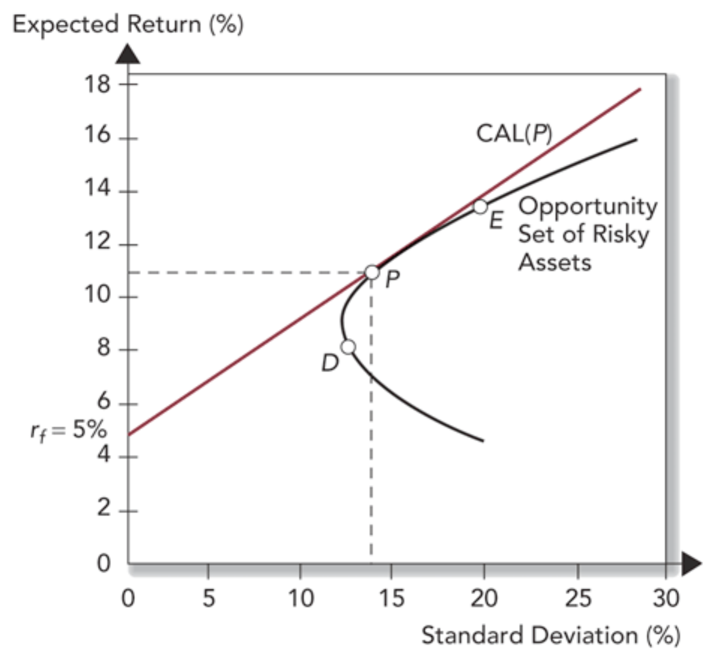

FIGURE 7.7 The opportunity set of the debt and equity funds with the optimal CAL and the optimal risky portfolio

3. Optimal Risky Portfolios

3.2. Efficient Portfolios: Two Risky Assets

- Two Risky Assets and a Risk Free Asset
  - Optimization problem: maximize the slope of the CAL for any possible portfolio, p
  - The objective function is the slope:

$$S_P = \frac{E(r_P) - r_f}{\sigma_P}$$

  - Which is Sharpe's Ratio

3. Optimal Risky Portfolios

3.2. Efficient Portfolios: Two Risky Assets

- Two Risky Assets and a Risk Free Asset
  - Optimal risky portfolio

$$w_D = \frac{\left[E\left(R_D\right)-r_f\right]\sigma_E^2 - \left[E\left(R_E\right)-r_f\right]Cov\left(r_D, r_E\right)}{\left[E\left(R_D\right)-r_f\right]\sigma_E^2 + \left[E\left(R_E\right)-r_f\right]\sigma_D^2 - \left[E\left(R_D\right)-r_f + E\left(R_E\right)-r_f\right]Cov\left(r_D, r_E\right)}$$

$$w_E = 1 - w_D$$

3. Optimal Risky Portfolios

3.2. Efficient Portfolios: Two Risky Assets

- Two Risky Assets and a Risk Free Asset

Example (optimal risky portfolio)

$$w_D = \frac{(8-5)400-(13-5)72}{(8-5)400+(13-5)144-(8-5+13-5)72} = 0.40$$

$$w_E = 1-0.40 = 0.60$$

$$E(r_P) = (0.4\times 8)+(0.6\times 13) = 11\%$$

$$\sigma_p = \left[(0.4^2\times 144)+(0.6^2\times 400)+(2\times 0.4\times 0.6\times 72)\right]^{\frac{1}{2}} = 14.2\%$$

$$S_p = \frac{11-5}{14.2} = 0.42$$

3. Optimal Risky Portfolios

3.2. Efficient Portfolios: Two Risky Assets

- Optimal Portfolio

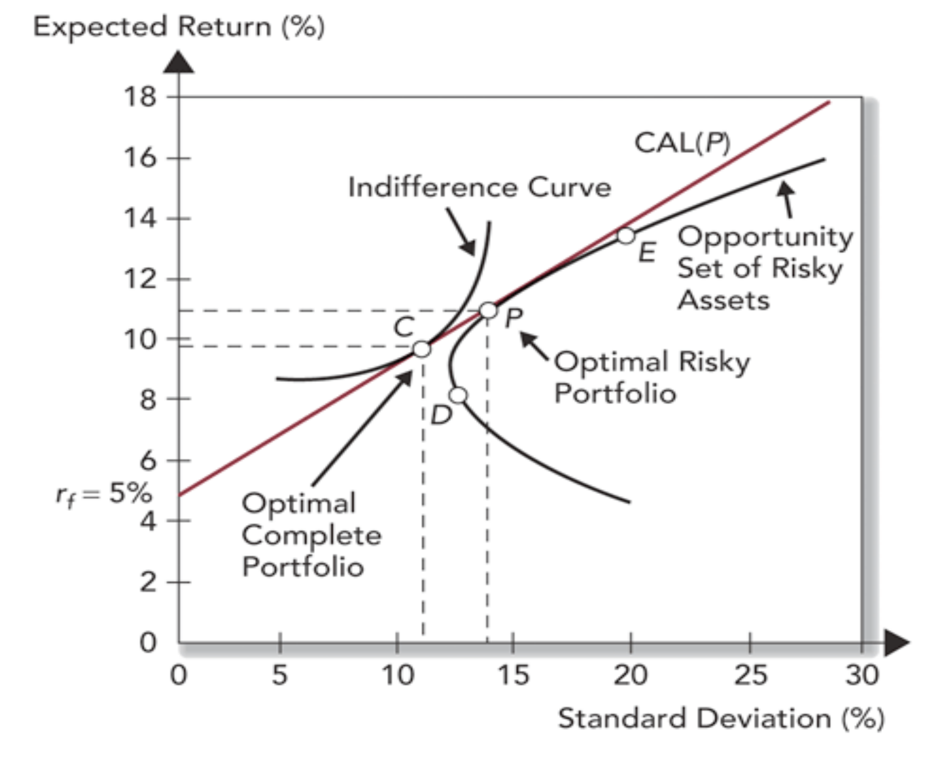

FIGURE 7.8 Determination of the optimal complete portfolio

3. Optimal Risky Portfolios

3.2. Efficient Portfolios: Two Risky Assets

- Optimal Portfolio

Example (optimal risky portfolio)

$$w_C = \frac{E(r_p) - r_f}{A\sigma_p^2} = \frac{0.11 - 0.05}{4 \times 0.142^2} = 0.7439$$

$$w_f = 1 - w_C = 1 - 0.7439 = 0.2561$$

$$w_{C_D} = w_C w_D = 0.7439 \times 0.40 = 0.2976$$

$$W_{C_E} = w_C W_E = 0.7439 \times 0.60 = 0.4463$$

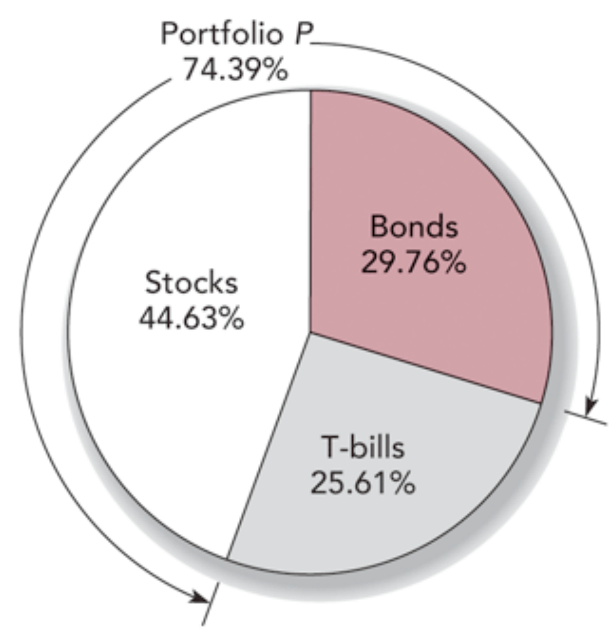

FIGURE 7.9 The proportions of the optimal complete portfolio

3. Optimal Risky Portfolios

3.3. Efficient Portfolios: N Risky Assets

- Security Selection
  - First step is to determine the risk-return opportunities available
  - All portfolios that lie on the minimum-variance frontier from the global minimum-variance portfolio and upward provide the best risk-return combinations

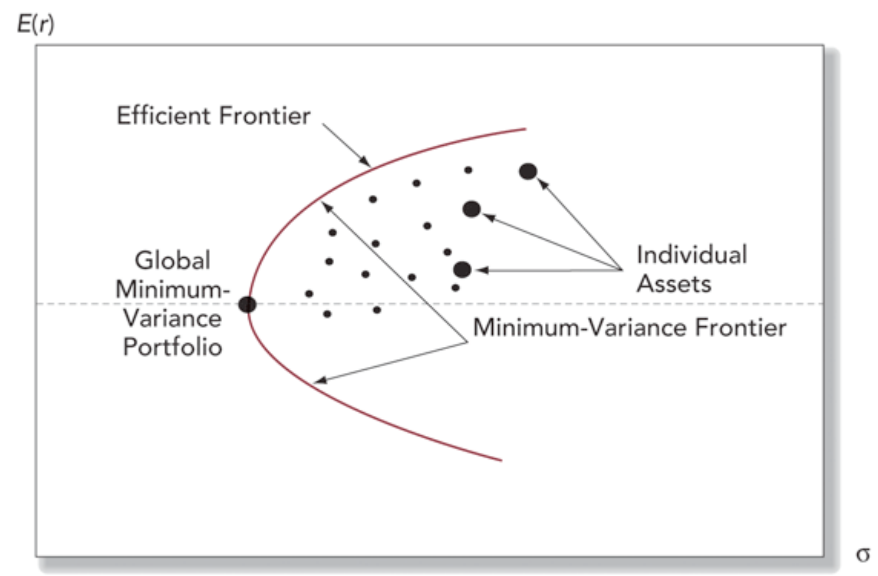

FIGURE 7.10 The minimum-variance frontier of risky assets

3. Optimal Risky Portfolios

3.3. Efficient Portfolios: N Risky Assets

- Efficient Portfolio Set

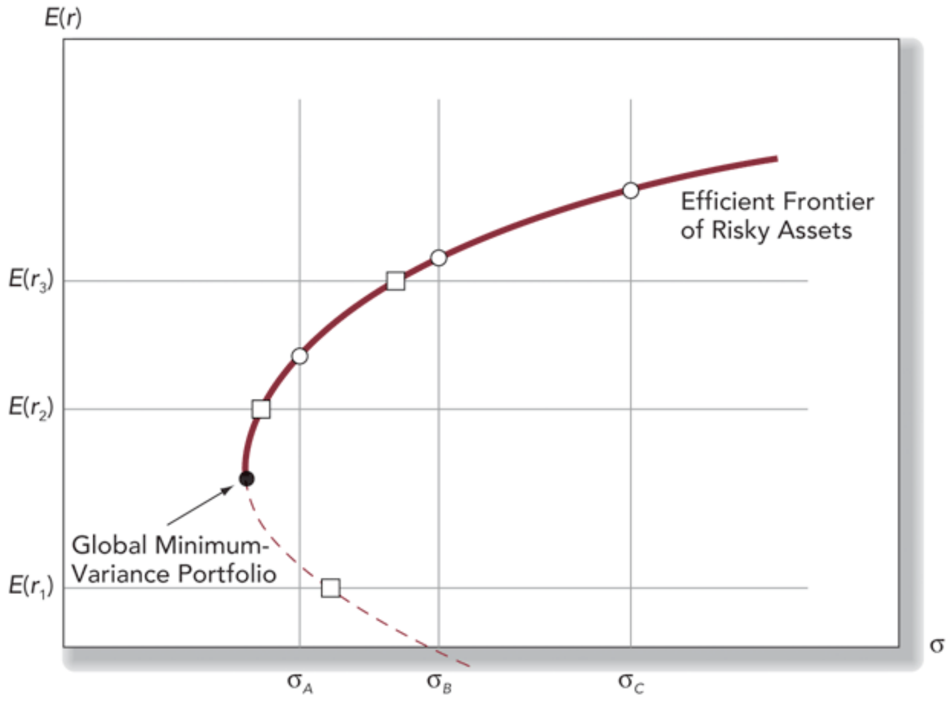

FIGURE 7.12 The efficient portfolio set

3.3. Efficient Portfolios: N Risky Assets

- Security Selection
  - We now search for the CAL with the highest reward-to-variability ratio
  - Finally, the individual chooses the appropriate mix between the optimal risky portfolio *P* and T-bills

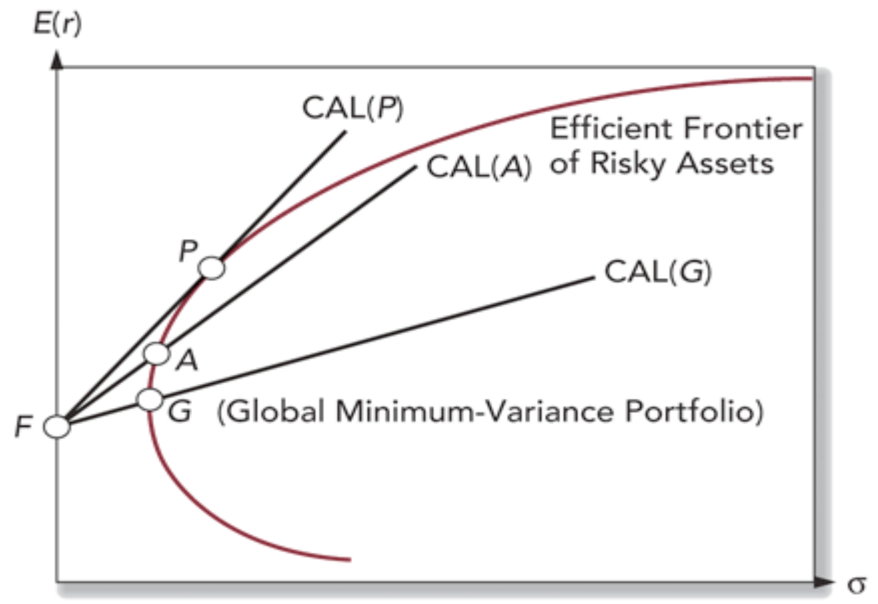

FIGURE 7.13 Capital allocation lines with various portfolios from the efficient set

3. Optimal Risky Portfolios

3.3. Efficient Portfolios: N Risky Assets

- Security Selection
  - Finally, the individual chooses the appropriate mix between the optimal risky portfolio *P* and T-bills

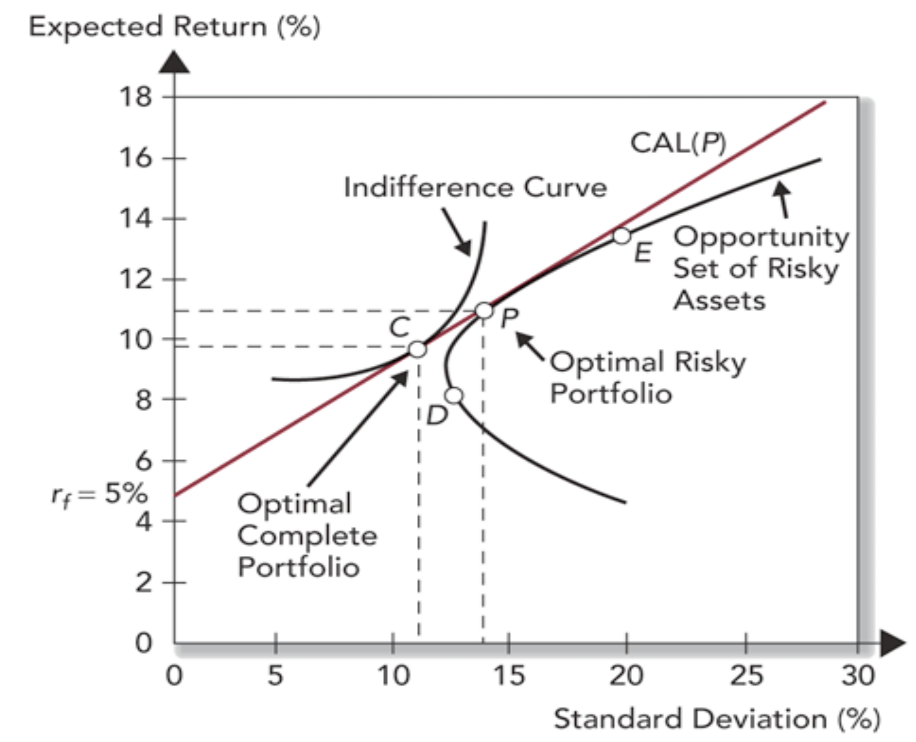

FIGURE 7.8 Determination of the optimal complete portfolio

3.3. Efficient Portfolios: N Risky Assets

- Capital Allocation and the Separation Property
  - The separation property tells us that the portfolio choice problem may be separated into two independent tasks
    - Determination of the optimal risky portfolio is purely technical
    - Allocation of the complete portfolio to T-bills versus the risky portfolio depends on personal preference

3.3. Efficient Portfolios: N Risky Assets

- The Power of Diversification
  - For n risky assets

$$\sigma_P^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j Cov(r_i, r_j) = \sum_{i=1}^{n} w_i \sigma_i^2 + \sum_{i=1}^{n}\sum_{\substack{j=1 \\ j\neq i}}^{n} w_i w_j Cov(r_i, r_j)$$

  - Consider the naïve diversification: $w_i = 1/n, \forall i$

$$\sigma_P^2 = \frac{1}{n}\sum\nolimits_{i=1}^{n} \frac{1}{n}\sigma_i^2 \quad + \sum\sum\nolimits_{j\neq i} \frac{1}{n^2} Cov(r_i; r_j)$$

3.3. Efficient Portfolios: N Risky Assets

- If we define the average variance and average covariance of the securities as:

$$\overline{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}\sigma_i^2 \quad \wedge \quad \overline{Cov} = \frac{1}{n(n-1)}\sum_{\substack{j=1 \\ j\neq i}}^{n}\sum_{i=1}^{n} Cov(r_i, r_j)$$

- We can then express portfolio variance as:

$$\sigma_P^2 = \frac{1}{n}\overline{\sigma}^2 + \frac{n-1}{n}\overline{Cov}$$

3. Optimal Risky Portfolios

3.3. Efficient Portfolios: N Risky Assets

- The Power of Diversification
  - Or, if we assume common standard deviation, σ, and common correlation coefficient, ρ

$$\sigma_P^2 = \frac{1}{n}\sigma^2 + \frac{n-1}{n}Cov$$

$$\sigma_P^2 = \frac{1}{n}\sigma^2 \quad + \frac{n-1}{n}\rho\sigma^2$$

3.3. Efficient Portfolios: N Risky Assets

- The Power of Diversification

TABLE 7.4

Risk reduction of equally weighted portfolios in correlated and uncorrelated universes

		ρ = 0		ρ = .4	
Universe Size *n*	Portfolio Weights *w* = 1/*n* (%)	Standard Deviation (%)	Reduction in σ	Standard Deviation (%)	Reduction in σ
1	100	50.00	14.64	50.00	8.17
2	50	35.36		41.83	
5	20	22.36	1.95	36.06	0.70
6	16.67	20.41		35.36	
10	10	15.81	0.73	33.91	0.20
11	9.09	15.08		33.71	
20	5	11.18	0.27	32.79	0.06
21	4.76	10.91		32.73	
100	1	5.00	0.02	31.86	0.00
101	0.99	4.98		31.86	