Part II

Portfolio Theory

4. Single and Multifactor Models

Financial Markets and Investments

Adapted from Bodie, Kane and Marcus' slides, McGraw-Hill/Irwin

Advantages of the Single Factor Model

- Reduces the number of inputs for diversification
- Easier for security analysts to specialize

4. Single and Multifactor Models

4.1. Single Factor Model

$$r_i = E(r_i) + \beta_i m + e_i$$

β_i : index of a securities' particular return to the factor, sensitivity of *i* to factor *m* ;

m : Unanticipated movement related to security returns, some common macroeconomic factor;

e_i : error; *m* and e_i are uncorrelated;

Assumption: a broad market index like the S&P 500 is the common factor.

4. Single and Multifactor Models

4.1. Single Factor Model

- *M*: Market index, with excess return
  - $R_M = r_M - r_f; \quad \sigma_M$
- Excess return of security *i*: $R_i = r_i - r_f$
- Regression Equation:

$$R_t(t) = \alpha_i + \beta_t R_M(t) + e_i(t)$$

- We collect historical info for R_M and R_i
- Expected return-beta relationship:

$$E(R_i) = \alpha_i + \beta_i E(R_M)$$

4. Single and Multifactor Models

4.1. Single Factor Model

- Risk and covariance:
  - Total risk = Systematic risk + Firm-specific risk:

$$\sigma_i^2 = \beta_i^2 \sigma_M^2 + \sigma^2(e_i)$$

  - Covariance = product of betas x market index risk:

$$Cov(r_i, r_j) = \beta_i \beta_j \sigma_M^2$$

  - Correlation = product of correlations with the market index

$$Corr(r_i, r_j) = \frac{\beta_i \beta_j \sigma_M^2}{\sigma_i \sigma_j} = \frac{\beta_i \sigma_M^2 \beta_j \sigma_M^2}{\sigma_i \sigma_M \sigma_j \sigma_M} = Corr(r_i, r_M) x Corr(r_j, r_M)$$

4. Single and Multifactor Models

4.1. Single Factor Model

- Suppose an equally weighted portfolio of *n* secutities
- **Portfolio**'s variance:

$$\sigma_P^2 = \beta_P^2 \sigma_M^2 + \sigma^2(e_P)$$

- Variance of the equally weighted portfolio of firm-specific components:

$$\sigma^2(e_P) = \sum_{i=1}^{n} \left(\frac{1}{n}\right)^2 \sigma^2(e_i) = \frac{1}{n}\bar{\sigma}^2(e)$$

- When *n* gets large, $\sigma^2(e_P)$ becomes negligible

4. Single and Multifactor Models

4.1. Single Factor Model

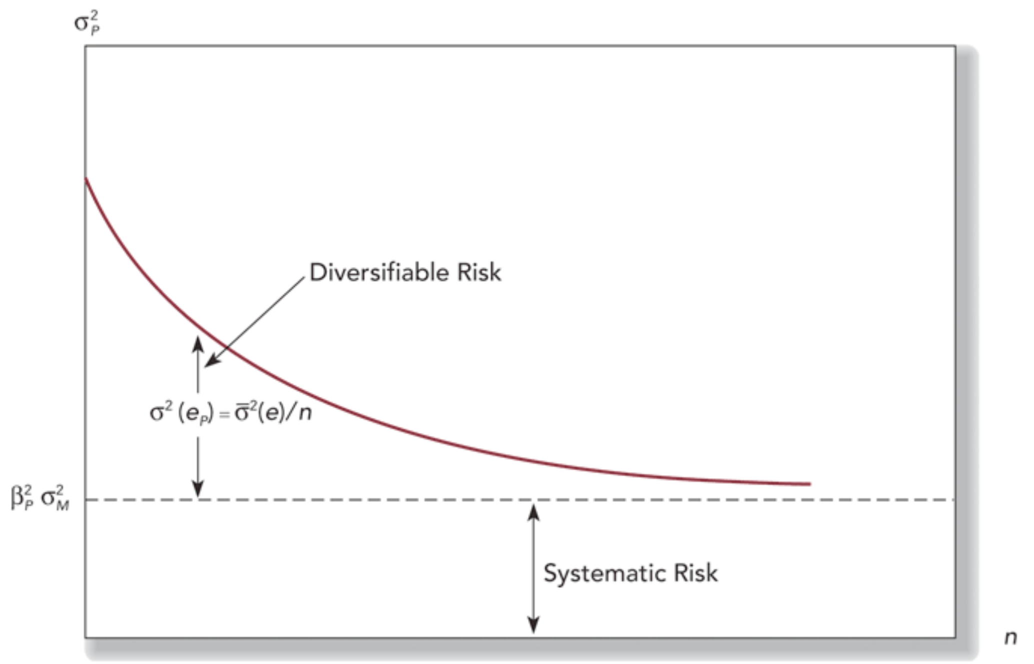

FIGURE 8.1 The variance of an equally weighted portfolio with risk coefficient β_P in the single-factor economy

4.2. Estimating the Single Factor Model

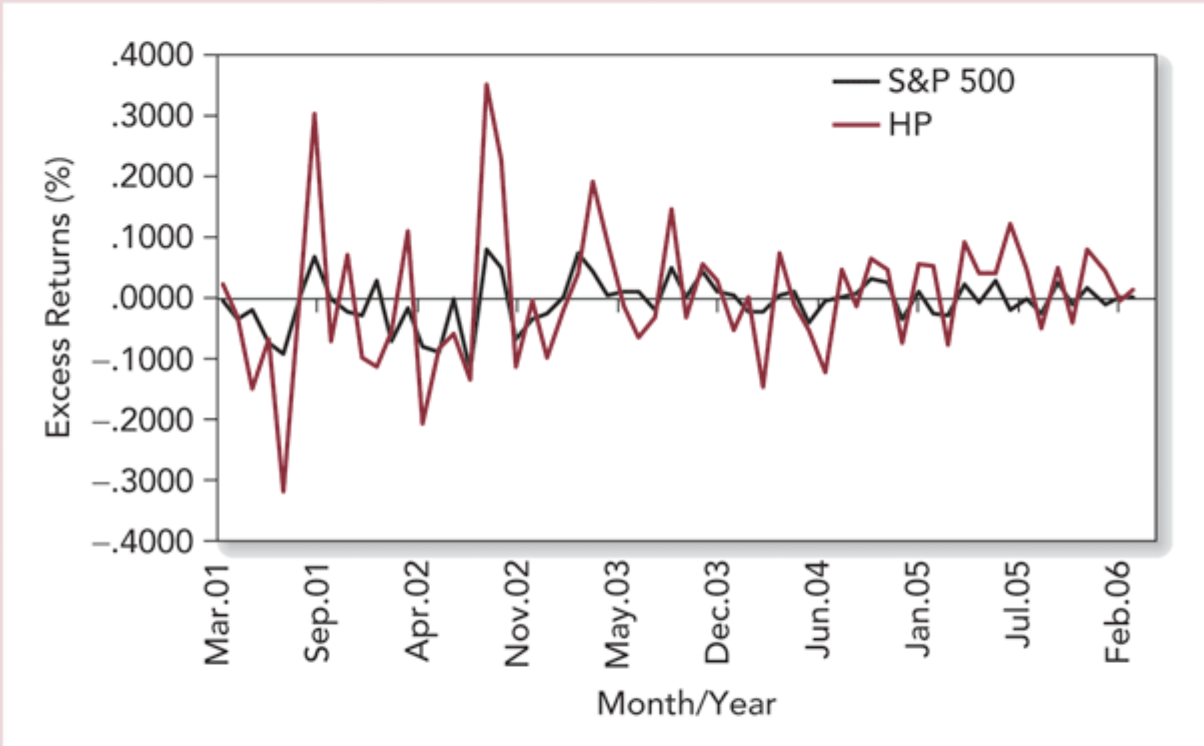

FIGURE 8.2 Excess returns on HP and S&P 500 for April 2001 to March 2006

4. Single and Multifactor Models

4.2. Estimating the Single Factor Model

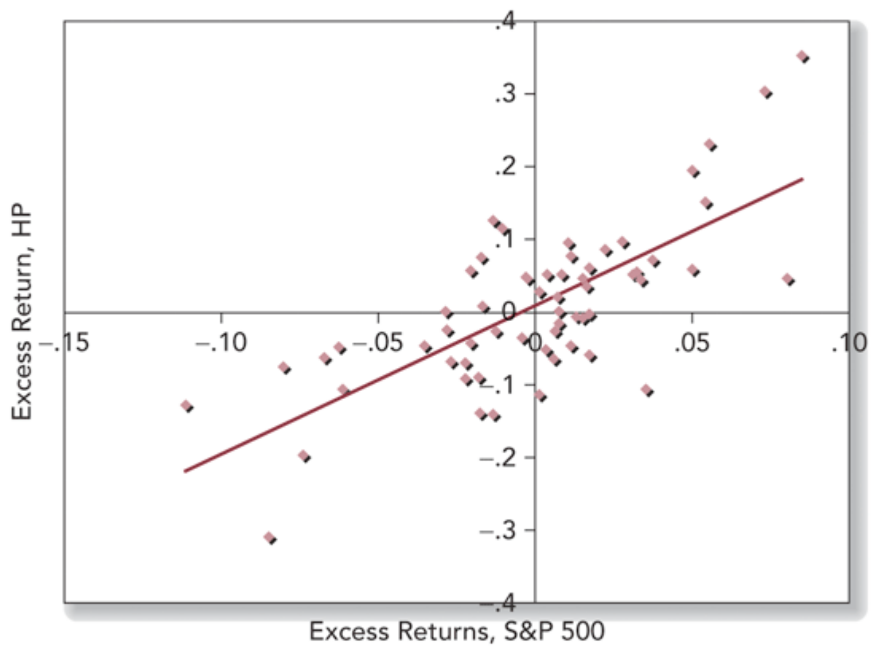

FIGURE 8.3 Scatter diagram of HP, the S&P 500, and the security characteristic line (SCL) for HP

4.2. Estimating the Single Factor Model

TABLE 8.1

Excel output: Regression statistics for the SCL of Hewlett-Packard

Regression Statistics				
Multiple *R*	.7238			
R-square	.5239			
Adjusted *R*-square	.5157			
Standard error	.0767			
Observations	60			
ANOVA				
	df	**SS**	**MS**	
Regression	1	.3752	.3752	
Residual	58	.3410	.0059	
Total	59	.7162		
	Coefficients	**Standard Error**	***t-Stat***	***p-Value***
Intercept	0.0086	.0099	0.8719	.3868
S&P 500	2.0348	.2547	7.9888	.0000

4. Single and Multifactor Models

4.3. Alpha and Security Analysis

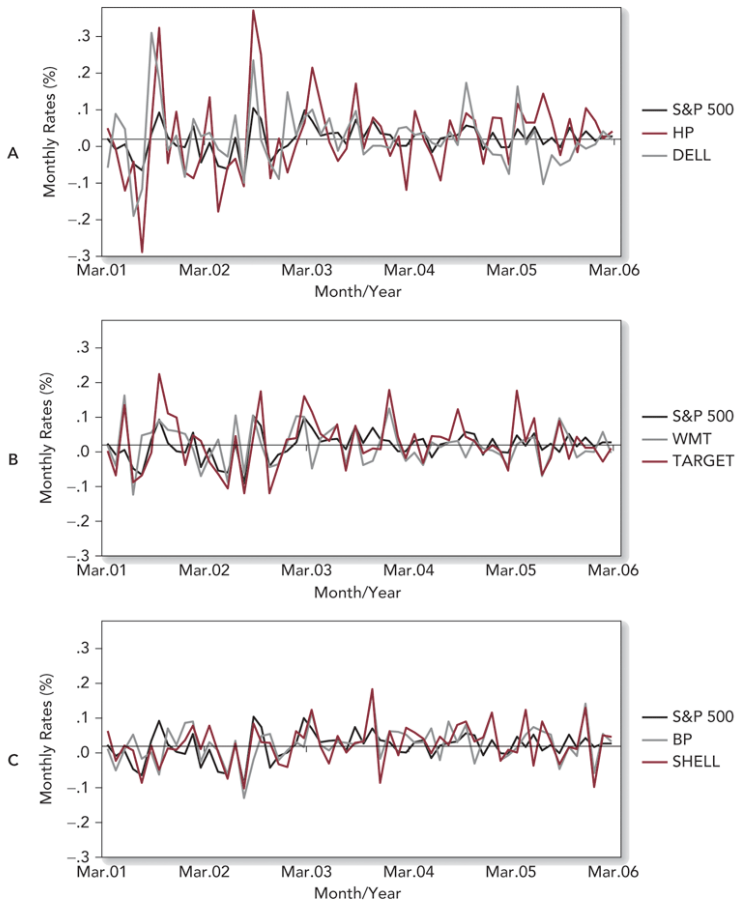

FIGURE 8.4 Excess returns on portfolio assets

4.3. Alpha and Security Analysis

- Macroeconomic analysis is used to estimate the risk premium and risk of the market index
- Statistical analysis is used to estimate the beta coefficients of all securities and their residual variances, $\sigma^2(e_i)$
- Developed from security analysis

4.3. Alpha and Security Analysis

- The market-driven expected return is conditional on information common to all securities
- Security-specific expected return forecasts are derived from various security-valuation models
  - The alpha value distills the incremental risk premium attributable to private information
- Helps determine whether security is a good or bad buy

Single-Index Model Input List

- Risk premium on the S&P 500 portfolio
- Estimate of the SD of the S&P 500 portfolio
- *n* sets of estimates of
  - Beta coefficient
  - Stock residual variances
  - Alpha values

Optimal Risky Portfolio of the Single-Index Model

- Maximize the Sharpe ratio
  - Expected return, SD, and Sharpe ratio:

$$E(R_P) = \alpha_P + E(R_M)\beta_P = \sum_{i=1}^{n+1} w_i\alpha_i + E(R_M)\sum_{i=1}^{n+1} w_i\beta_i$$

$$\sigma_P = \left[\beta_P^2\sigma_M^2 + \sigma^2(e_P)\right]^{\frac{1}{2}} = \left[\sigma_M^2\left(\sum_{i=1}^{n+1} w_i\beta_i\right)^2 + \sum_{i=1}^{n+1} w_i^2\sigma^2(e_i)\right]^{\frac{1}{2}}$$

$$S_P = \frac{E(R_P)}{\sigma_P}$$

4.4. Portfolio Construction

- Combination of:
  - Active portfolio denoted by *A*
  - Market-index portfolio, the (*n*+1)th asset which we call the passive portfolio and denote by *M*
  - Modification of active portfolio position:

$$w_A^* = \frac{w_A^0}{1 + (1 - \beta_A) w_A^0}$$

  - When $\beta_A = 1, w_A^* = w_A^0$

4.4. Portfolio Construction

- The Sharpe ratio of an optimally constructed risky portfolio will exceed that of the index portfolio (the passive strategy):

$$S_P^2 = S_M^2 + \left[\frac{\alpha_A}{\sigma(e_A)}\right]^2$$

4. Single and Multifactor Models

4.4. Portfolio Construction

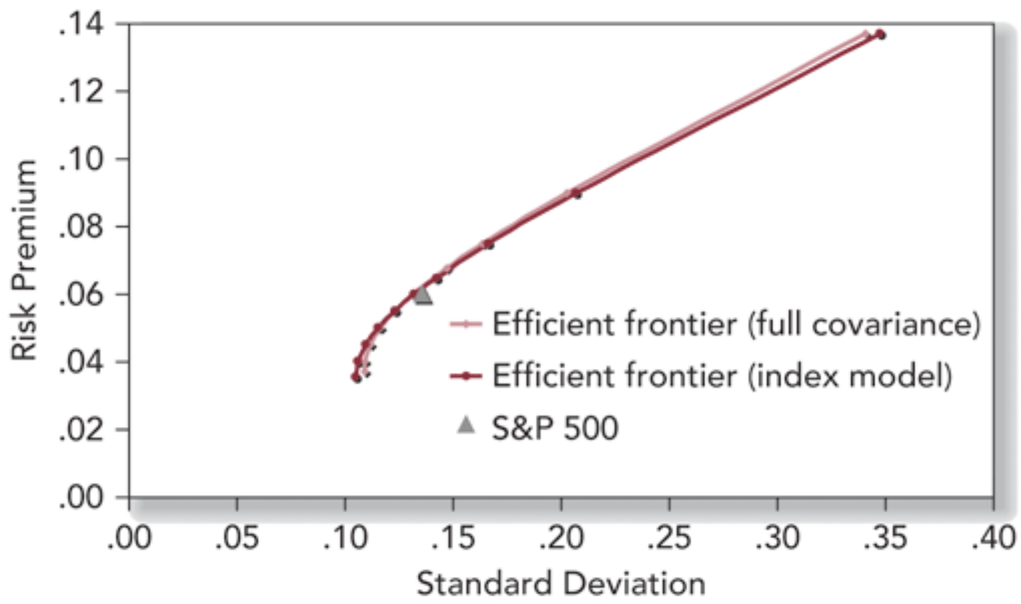

FIGURE 8.5 Efficient frontiers with the index model and full-covariance matrix

4.4. Portfolio Construction

TABLE 8.2

Comparison of portfolios from the single-index and full-covariance models

	Global Minimum Variance Portfolio		Optimal Portfolio	
	Full-Covariance Model	**Index Model**	**Full-Covariance Model**	**Index Model**
Mean	.0371	.0354	.0677	.0649
SD	.1089	.1052	.1471	.1423
Sharpe ratio	.3409	.3370	.4605	.4558
Portfolio Weights				
S&P 500	.88	.83	.75	.83
HP	−.11	−.17	.10	.07
DELL	−.01	−.05	−.04	−.06
WMT	.23	.14	−.03	−.05
TARGET	−.18	−.08	.10	.06
BP	.22	.20	.25	.13
SHELL	−.02	.12	−.12	.03

4.4. Portfolio Construction

Ticker Symbol	Security Nam		2004/12 Close Price	Beta	Alpha	R-Sqr	Resid Std Dev-n	Std Error Beta	Std Error Alpha	Adjusted Beta	Number of Observ
HTBK	HERITAGE COMM CORP		19.020	0.23	0.72	0.01	6.86	0.19	0.89	0.49	60
HPC	HERCULES INC		14.850	0.78	−0.09	0.07	12.13	0.34	1.57	0.85	60
HFWA	HERITAGE FINL CORP WASH		22.120	0.09	1.69	−0.01	4.27	0.12	0.55	0.40	60
HRLY	HERLEY INDS INC		20.340	−0.04	1.66	−0.02	10.37	0.29	1.34	0.31	60
HT	HERSHA HOSPITALITY TR	PRIORITY A SHS	11.450	0.46	1.67	0.12	5.62	0.16	0.73	0.64	60
HSY	HERSHEY FOODS CORP		55.540	−0.21	1.66	0.00	7.72	0.21	1.00	0.20	60
HSKA	HESKA CORP		1.169	1.87	3.88	0.06	31.26	0.86	4.04	1.58	60
HPQ	HEWLETT PACKARD CO		20.970	1.76	−0.45	0.40	10.05	0.28	1.30	1.50	60
HXL	HEXCEL CORP NEW		14.500	0.85	4.08	0.02	21.63	0.60	2.80	0.90	60
HIFN	HI/FN INC		9.220	2.33	0.88	0.21	20.55	0.57	2.66	1.88	60
HIBB	HIBBETT SPORTING GOODS		26.610	1.03	4.05	0.11	13.03	0.36	1.68	1.02	60
HIB	HIBERNIA CORP	CLASS A	29.510	0.59	2.08	0.14	6.53	0.18	0.84	0.73	60
HICK A	HICKOK INC	CLASS A	7.500	0.29	2.35	−0.01	19.21	0.53	2.48	0.53	60
HTCO	HICKORY TECH CORP		10.690	0.13	−0.02	−0.01	10.74	0.30	1.39	0.42	60
HSVL Y	HIGHVELD STL & VANADIUM ADR		8.200	0.34	2.64	0.00	14.42	0.40	1.86	0.56	60
HIW	HIGHWOODS PROPERTIES IN		27.700	0.10	0.45	–0.01	5.70	0.16	0.74	0.40	60

TABLE 8.3

Merrill Lynch, Pierce, Fenner & Smith, Inc.: Market sensitivity statistics*

*Based on S&P 500 index using straight regression.

4.4. Portfolio Construction

Industry	Beta	Adjustment Factor
Agriculture	0.99	–.140
Drugs and medicine	1.14	–.099
Telephone	0.75	–.288
Energy utilities	0.60	–.237
Gold	0.36	–.827
Construction	1.27	.062
Air transport	1.80	.348
Trucking	1.31	.098
Consumer durables	1.44	.132

TABLE 8.4

Industry betas and adjustment factors

4.5. Multifactor Models

- Returns on a security come from two sources
  - Common macro-economic factor
  - Firm specific events
- Possible common macro-economic factors
  - Gross Domestic Product Growth
  - Interest Rates

4.5. Multifactor Models

- Use more than one factor in addition to market return
  - Examples include gross domestic product, expected inflation, interest rates etc.
  - Estimate a beta or factor loading for each factor using multiple regression.

4. Single and Multifactor Models

4.5. Multifactor Models

$$r_i = E(r_i) + \beta_{i\,GDP}\, GDP + \beta_{i\,IR}\, IR + e_i$$

r_i = Return for security *i*

$\beta_{i\,GDP}$ = Factor sensitivity for GDP

$\beta_{i\,IR}$ = Factor sensitivity for Interest Rate

e_i = Firm specific events

4.5. Multifactor Models

Multifactor SML Models

$$E(r) = r_f + \beta_{i_{GDP}} RP_{GDP} + \beta_{i_{IR}} RP_{IR}$$

$\beta_{i_{GDP}}$ = Factor sensitivity for GDP

RP_{GDP} = Risk premium for GDP

$\beta_{i_{IR}}$ = Factor sensitivity for Interest Rate

RP_{IR} = Risk premium for Interest Rate