Lévy processes and applications - Theoretical overview

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Lévy-Khintchine formula

- $(\Omega, \mathcal{F}, P)$.
- A Lévy Process $X = (X(t), t \geq 0)$ is essentially a stochastic process with stationary and indepedent increments.
- Key-formula: the Lévy-Khintchine formula:

$$E\left[e^{i(u,X(t))}\right] = e^{t\eta(u)}$$

where

$$\eta(u) = i(b,u) - \frac{1}{2}(u,Au) + \int_{\mathbb{R}^d - \{0\}} \left[e^{i(u,y)} - 1 - i(u,y)\chi_{(0<|y|<1)}(y)\right]\nu(dy),$$

$b \in \mathbb{R}^d$, A is a $d \times d$ positive definite symmetric matrix and a ν is a Lévy measure on $\mathbb{R}^d - \{0\}$ such that, for all $u \in \mathbb{R}^d$:

$$\int_{\mathbb{R}^d - \{0\}} \left(|y|^2 \wedge 1\right)\nu(dy) < \infty.$$

Lévy-Khintchine formula

- Let $A = \nu = 0$. Then
$$E\left[e^{i(u,X(t))}\right] = e^{it(u,b)}$$
and $X(t) = bt$ is a deterministic motion in a stright line (b is the velocity of the motion - drift)
- Let $A \neq 0$ and $\nu = 0$. Then
$$E\left[e^{i(u,X(t))}\right] = \exp\left[t\left[i(b,u) - \frac{1}{2}(u,Au)\right]\right],$$
which is the characteristic of a Gaussian r.v. with mean tb and covariance matrix tA.In fact $X(t)$ is a Brownian motion with drift.

Lévy-Khintchine formula

- Let ν be a finite measure ($\lambda = \int_{\mathbb{R}^d} \nu(dx) < \infty$). Then
$$\eta(u) = i(b',u) - \frac{1}{2}(u,Au) + \int_{\mathbb{R}^d - \{0\}} \left[e^{i(u,y)} - 1\right]\nu(dy),$$
with $b' = b - \int_{0<|y|<1} y\nu(dy)$.
- If $\nu = \lambda\delta_h$ with $\lambda > 0$ then
$$X(t) = b't + \sigma B(t) + N(t),$$
where $\sigma = \sqrt{A}$ (or $\sigma\sigma^T = A$) and $N(t)$ is a Poisson process of intensity λ with jumps of size $|h|$. Note that $E\left[e^{i(u,N(t))}\right] = \exp\left[\lambda t\left(e^{i(u,h)} - 1\right)\right]$

Lévy-Khintchine formula

- Let $\nu = \sum_{i=1}^{m} \lambda_i \delta_{h_i}$. Then

$$X(t) = b't + \sigma B(t) + N_1(t) + \cdots N_m(t),$$

where the $N_i's$ are independent Poisson processes (also independent of B), with N_i of intensity λ_i with jumps of size $|h_i|$.

- General case with finite ν corresponds to jump sizes in a continuum of possibilities (continuum of Poisson processes - jumps of arbitrary size).

Remarks

- Most subtle case: infinite measure case: $\int_{0<|y|<1} |y| \, \nu(dy) = \infty$ and $\int_{0<|y|<1} |y|^2 \, \nu(dy) < \infty$.
- Then $e^{i(u,y)} - 1$ may no longer be ν-integrable, but $e^{i(u,y)} - 1 - i(u,y)\chi_{(0<|y|<1)}(y)$ is allways ν-integrable
- intuition: ν has become so fine that an infinite number of small jumps is expected.
- When ν is finite we can write:

$$X(t) = bt + \sigma B(t) + \sum_{0 \leq s \leq t} \Delta X(s),$$

- $\Delta X(s)$ is the jump at time s.

Lévy-Itô decomposition

- For each Borel set $A \in \mathbb{R}^d - \{0\}$, let $N(t, A) = \#\{0 \le s \le t : \Delta X(s) \in A\}$
- Fix t and A: then $N(t, A)$ is a r.v.
- Fix $\omega \in \Omega$ and t: then $N(t, \cdot)(\omega)$ is a measure
- Fix A. Then $\{N(t, A), t \ge 0\}$ is a Poisson process with intensity $\nu(A)$ with

$$\sum_{0 \le s \le t} \Delta X(s) = \int_{\mathbb{R}-\{0\}} x N(t, dx)$$

- In the case of infinite measure ν, we have the Lévy-Itô decomposition:

$$X(t) = bt + \sigma B(t) + \int_{0<|x|<1} x\left[N(t, dx) - t\nu(dx)\right] + \\ + \int_{|x|\ge 1} x N(t, dx)$$

Applebaum, D. (2004). Lévy Processes and Stochastic Caculus. Cambridge University Press. - (Overview and chapter 1)

Papantaloleon, A. An Introduction to Lévy Processes with Applications in Finance. arXiv:0804.0482v2.