Lévy processes and applications - Infinite divisibility

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Infinite divisibility

- $(\Omega, \mathcal{F}, P)$.
- Law of X: $p_X(A) = P(X \in A)$ for $A \in \mathcal{F}$.
- If X and Y are independent random variables then the law of $X + Y$ is the convolution of measures:

$$p_{X+Y} = p_X * p_Y$$

- Convolution of probab. measures:

$$(p_X * p_Y)(A) = \int_{\mathbb{R}^d} p_X(A - x)\, p_Y(dx)$$

- Equivalently:

$$\int_{\mathbb{R}^d} g(y)\,(p_X * p_Y)(dy) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(x + y)\, p_X(dx)\, p_Y(dy)$$

  for all g bounded measurable function.
- If X and Y are independent with densities f_X and f_Y then

$$f_{X+Y}(x) = (f_X * f_Y)(x) = \int_{\mathbb{R}^d} f_X(x - y)\, f_Y(y)\, dy$$

Infinite divisibility

- Characteristic function of X is $\phi_X : \mathbb{R}^d \to \mathbb{C}$ defined by

$$\phi_X(u) = E\left[e^{i(u,X)}\right] = \int_{\mathbb{R}^d} e^{i(u,x)} p_X(dx).$$

- Let μ be a probability measure on $\mathbb{R}^d$
- Properties of the characteristic function:
  1. $\phi_\mu(0) = 1$
  2. ϕ_μ is positive definite: $\sum_{i,j} c_i \overline{c_j} \phi_\mu(u_i - u_j) \geq 0$ for all $c_i \in \mathbb{C}$, $u_i \in \mathbb{R}^d$, $1 \leq i, j \leq n$, $n \in \mathbb{N}$.
  3. $|\phi_\mu(u)| \leq 1$
  4. ϕ_μ is unformly continuous
- Moreover, $\mu \to \phi_\mu$ is injective
- Bochner Theorem: If $\phi : \mathbb{R}^d \to \mathbb{C}$ satisfies 1. and 2. and is continuous at $u = 0$, then it is a characteristic function.
- Exercise 1: Show that $|\phi_\mu(u)| \leq 1$.

Infinite divisibility

- A Lévy process incorporates "infinite divisibility".
- Notation: $\mu^{*n} = \mu * \mu * \cdots * \mu$

Definition

μ has a convolution nth root if exists a probab. meas. $\mu^{\frac{1}{n}}$ such that

$$\left(\mu^{\frac{1}{n}}\right)^{*n} = \mu.$$

Definition

μ is infinitely divisible if it has a convolution nth root for all $n \in \mathbb{N}$.

Infinite divisibility

Theorem

μ is infinitely divisible iff for all $n \in \mathbb{N}$, exists μ_n with charact. func. ϕ_n:

$$\phi_\mu(u) = (\phi_n(u))^n$$

for all $u \in \mathbb{R}^d$.

Proof: ($\Longrightarrow$) Take $\phi_n = \phi_{\mu^{1/n}}$
($\Longleftarrow$) For each n, by Fubini's theorem:

$$\begin{aligned}\phi_\mu(u) &= \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} e^{i(u, y_1 + y_2 + \cdots + y_n)} \mu_n(dy_1) \cdots \mu_n(dy_n) \\ &= \int_{\mathbb{R}^d} e^{i(u,y)} (\mu_n)^{*n}(dy)\end{aligned}$$

and since ϕ determines μ uniquely then $\mu = (\mu_n)^{*n}$ and is infinitely divisible. ■

Infinite divisibility

Proposition

Properties:

1. *μ and ν infinitely divisibles $\Longrightarrow \mu * \nu$ is inf. divis.*
2. *If $\{\mu_n, n \in \mathbb{N}\}$ are inf. divis. and $\mu_n \xrightarrow{w} \mu$ then μ is inf. div.*

- Remark: $\mu_n \xrightarrow{w} \mu$ means that μ_n converges weakly to μ, i.e.,

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f(x)\, \mu_n(dx) = \int_{\mathbb{R}^d} f(x)\, \mu(dx)$$

  for all $f \in C_b\left(\mathbb{R}^d\right)$ (bounded continuous functions).
- Exercise 2: Show that Property 1 holds

Infinite divisibility

Definition

A r.v. X is inf. divis. if its law p_X is inf. divis. This means that

$$X \stackrel{d}{=} Y_1^{(n)} + Y_2^{(n)} + \cdots + Y_n^{(n)},$$

where $Y_1^{(n)}, \ldots, Y_n^{(n)}$ are iid, for each $n \in \mathbb{N}$.

Proposition

The following are equivalent:

1. *X is inf. divis.*
2. *μ_X has a convolution nth root that is the law of a r.v., for each $n \in \mathbb{N}$*
3. *ϕ_X has an nth root that is the charac. funct. of a r.v., for each $n \in \mathbb{N}$*

Infinite divisibility

- Exercise 3: Prove the previous Proposition
- Exercise 4: Let $\alpha > 0$, $\beta > 0$. Show that the gamma-(α, β) distribution

$$\mu_{\alpha,\beta}(dx) = \frac{\alpha^\beta}{\Gamma(\beta)} x^{\beta-1} e^{-\alpha x} dx, \quad \text{with } x > 0$$

is an infinitely-divisible distribution.

Infinite divisibility - Examples

- In each example, we will find iid $Y_1^{(n)}, \ldots, Y_n^{(n)}$ such that $X \overset{d}{=} Y_1^{(n)} + Y_2^{(n)} + \cdots + Y_n^{(n)}$.

Example

(Gaussian random variable) Let $X = (X_1, X_2, \ldots, X_d)$ be Gaussian random vector, with density:

$$f(x) = \frac{1}{(2\pi)^{\frac{d}{2}} \sqrt{\det(A)}} \exp\left(-\frac{1}{2}\left(x - m, A^{-1}(x - m)\right)\right), \quad x \in \mathbb{R}^d.$$

$X \sim N(m, A)$, where A is a $d \times d$ strictly positive-definite symmetric covariance matrix: $A = E\left[(X - m)(X - m)^T\right]$.
We can show (see probability book) that

$$\phi_X(u) = \exp\left(i(m, u) - \frac{1}{2}(u, Au)\right).$$

Infinite divisibility - Examples

Example

(continued) Therefore:

$$(\phi_X(u))^{\frac{1}{n}} = \exp\left(i\left(\frac{m}{n}, u\right) - \frac{1}{2}\left(u, \frac{1}{n}Au\right)\right).$$

and X is inf. divis. with $X \overset{d}{=} Y_1^{(n)} + Y_2^{(n)} + \cdots + Y_n^{(n)}$ and

$$Y_j^{(n)} \sim N\left(\frac{m}{n}, \frac{1}{n}A\right).$$

Infinite divisibility - Examples

Example

(Poisson r.v.) Let $d = 1$ and $X : \Omega \to \mathbb{N}_0$ with $X \sim Po(\lambda)$, i.e.

$$P(X = n) = \frac{\lambda^n}{n!} e^{-\lambda}.$$

It is well known that $E[X] = Var[X] = \lambda$ and it is easy to verify that

$$\phi_X(u) = \exp\left[\lambda\left(e^{iu} - 1\right)\right].$$

Therefore

$$(\phi_X(u))^{\frac{1}{n}} = \exp\left[\frac{\lambda}{n}\left(e^{iu} - 1\right)\right].$$

and X is inf. divis.with $X \stackrel{d}{=} Y_1^{(n)} + Y_2^{(n)} + \cdots + Y_n^{(n)}$ and

$$Y_j^{(n)} \sim Po\left(\frac{\lambda}{n}\right).$$

Example

(Compound Poisson r.v.) Let $\{Z(n), n \in \mathbb{N}\}$ be a sequence of iid r.v. with law μ_Z. Let $N \sim Po(\lambda)$ and independent of the $Z(n)'s$. Define

$$X = Z(1) + Z(2) + \cdots + Z(N).$$

Let us prove that, for each $u \in \mathbb{R}^d$,

$$\phi_X(u) = \exp\left[\int_{\mathbb{R}^d} \left(e^{i(u,y)} - 1\right) \lambda \mu_Z(dy)\right]. \quad (1)$$

Indeed, by conditioning, we have

$$\phi_X(u) = E\left[e^{i(u,X)}\right] = \sum_{n=0}^{\infty} E\left[e^{i(u,Z(1)+Z(2)+\cdots+Z(N))} | N = n\right] P[N = n]$$

$$= \sum_{n=0}^{\infty} E\left[e^{i(u,Z(1)+Z(2)+\cdots+Z(n))}\right] \frac{\lambda^n}{n!} e^{-\lambda}$$

$$= e^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda\phi_Z(u))^n}{n!} = \exp\left[\lambda\left(\phi_Z(u) - 1\right)\right].$$

Infinite divisibility - Examples

Example

(Continued) Therefore, with $\phi_Z(u) = \int_{\mathbb{R}^d} e^{i(u,y)} \mu_Z(dy)$, we obtain (1). We denote the Compound Poisson by $X \sim Po(\lambda, \mu_Z)$. We have

$$(\phi_X(u))^{\frac{1}{n}} = \exp\left[\frac{\lambda}{n}(\phi_Z(u) - 1)\right]$$

and X is inf. divis. with $X \stackrel{d}{=} Y_1^{(n)} + Y_2^{(n)} + \cdots + Y_n^{(n)}$ and

$$Y_j^{(n)} \sim Po\left(\frac{\lambda}{n}, \mu_Z\right).$$

- Exercise 5: Show that if $X \sim N(m, A)$, where A is a $d \times d$ strictly positive-definite symmetric covariance matrix: $A = E\left[(X - m)(X - m)^T\right]$ then $\phi_X(u) = \exp\left(i(m, u) - \frac{1}{2}(u, Au)\right)$
- Exercise 6: Let $d = 1$. Show that if $X \sim Po(\lambda)$ then $\phi_X(u) = \exp\left[\lambda\left(e^{iu} - 1\right)\right]$.

The Lévy measure

Definition

Let ν be a Borel measure defined on $\mathbb{R}^d - \{0\}$. We say that ν is a Lévy measure if

$$\int_{\mathbb{R}^d - \{0\}} \left(|y|^2 \wedge 1\right) \nu(dy) < \infty \qquad (2)$$

- Note that $|y|^2 \wedge \varepsilon \leq |y|^2 \wedge 1$ when $0 < \varepsilon \leq 1$. Therefore, by (2), we have that
$$\nu\left[(-\varepsilon, \varepsilon)^c\right] < \infty, \quad \text{for all } \varepsilon > 0.$$
- Note: Condition (2) is equivalent to
$$\int_{\mathbb{R}^d - \{0\}} \frac{|y|^2}{1 + |y|^2} \nu(dy) < \infty.$$
- Note: one can assume that $\nu(\{0\}) = 0$ and then ν is defined on $\mathbb{R}^d$.

Lévy-Khintchine formula

Theorem

(Lévy-Khintchine): A probab. measure μ on $\mathbb{R}^d$ is infinitely divisible if exists a vector $b \in \mathbb{R}^d$, a $d \times d$ positive definite symmetric matrix A and a Lévy measure ν on $\mathbb{R}^d - \{0\}$ such that, for all $u \in \mathbb{R}^d$:

$$\phi_\mu (u) = \exp \left\{ i (b,u) - \frac{1}{2} (u, Au) + \right. \\ \left. + \int_{\mathbb{R}^d - \{0\}} \left[e^{i(u,y)} - 1 - i (u,y) \chi_{\widehat{B}} (y) \right] \nu (dy) \right\}, \quad (3)$$

where $\widehat{B} = B_1 (0) = \left\{ y \in \mathbb{R}^d : |y| < 1 \right\}$.
Conversely, any mapping of the form (3) is the characteristic function of an inf. divis. probability measure on $\mathbb{R}^d$.

Lévy-Khintchine formula

- (b, A, ν) are the characteristics of the inf. divis. r.v. X.
- $\eta := \log (\phi_\mu)$ is the Lévy symbol or characteristic exponent or Lévy exponent:

$$\eta (u) = i (b,u) - \frac{1}{2} (u, Au) + \int_{\mathbb{R}^d - \{0\}} \left[e^{i(u,y)} - 1 - i (u,y) \chi_{\widehat{B}} (y) \right] \nu (dy).$$

- We will not prove the first part of the theorem (difficult)
- We will prove the second part

Proof (2nd part)

- We need to prove that the r.h.s of (3) is a characteristic function.
- i) Let $\{U(n), n \in \mathbb{N}\} \subset \mathbb{R}^d$ be a sequence of Borel sets such that $U(n) \searrow 0$ and define

$$\phi_n(u) = \exp\left\{ i\left(b - \int_{U(n)^c \cap \widehat{B}} y\nu(dy), u\right) - \frac{1}{2}(u, Au) + \right.$$
$$\left. + \int_{U(n)^c} \left(e^{i(u,y)} - 1\right)\nu(dy)\right\}.$$

ii) Clearly, ϕ_n is the convolution of a Normal dist. with an independent compound Poisson dist.. Therefore, by Proposition (properties of inf. divis.), it is infinit. divis.
iii) Clearly,

$$\phi_\mu(u) = \lim_{n\to\infty} \phi_n(u).$$

Proof (continued)

iv) In order to prove that ϕ_μ is a characteristic function , we will aplly Lévy's continuity theorem (see below) and therefore we only need to prove that $\psi_\mu(u)$ is continuous at zero, with

$$\psi_\mu(u) = \int_{\mathbb{R}^d - \{0\}} \left[e^{i(u,y)} - 1 - i(u,y)\chi_{\widehat{B}}(y)\right]\nu(dy)$$
$$= \int_{\widehat{B}} \left(e^{i(u,y)} - 1 - i(u,y)\right)\nu(dy) +$$
$$+ \int_{\widehat{B}^c} \left(e^{i(u,y)} - 1\right)\nu(dy).$$

v) By Taylor's theorem, the Cauchy-Schwarz inequality and dominated convergence, we have:

$$|\psi_\mu(u)| \leq \frac{1}{2}\int_{\widehat{B}} |(u,y)|^2 \nu(dy) + \int_{\widehat{B}^c} \left|e^{i(u,y)} - 1\right|\nu(dy)$$
$$\leq \frac{|u|^2}{2}\int_{\widehat{B}} |y|^2 \nu(dy) + \int_{\widehat{B}^c} \left|e^{i(u,y)} - 1\right|\nu(dy) \to 0 \text{ as } u \to 0.$$

vi) It is now easy to verify directly that μ is infin. divis. ■

Remarks

- This techique of taking the limits of sequences composed of sums of gaussians with independent compound Poissons is very important.
- The cut-off function $c(y) = y\chi_{\widehat{B}}$ in (3) could be replaced by other $c(y)$ such that $e^{i(u,y)} - 1 - i(u, c(y))$ is an ν-integrable function for each $u \in \mathbb{R}^d$. For instance, we could have $c(y) = \frac{y}{1+|y|^2}$.
- Gaussian case: $b = m$ (mean), $A =$covariance matrix, $\nu = 0$.
- Possion case: $b = 0$, $A = 0$, $\nu = \lambda\delta_1$
- Compound Poisson case: $b = 0$, $A = 0$, $\nu = \lambda\mu$, $\lambda > 0$ and μ a probab. measure on $\mathbb{R}^d$

Remarks

- All infinitely divisible distributions can be constructed as weak limits of convolutions of Gaussians with independent Poisson processes. In fact, they can be obtained as weak limits of Compound Poissons only.

Theorem

Any infinitely divisible probability measure can be obtained as the weak limit of a sequence of compound Poisson distributions.

Proof: If ϕ is infinitely divisible then $\phi^{\frac{1}{n}}$ is the charact. function of $\mu^{\frac{1}{n}}$ and

$$\phi_n(u) = \exp\left\{n\left[\phi^{\frac{1}{n}}(u) - 1\right]\right\}$$

is the charact. function of a compound Poisson. Moreover,

$$\phi_n(u) = \exp\left\{n\left[e^{\frac{1}{n}\log(\phi(u))} - 1\right]\right\} =$$

$$= \exp\left\{\log(\phi(u)) + n\, o\left(\frac{1}{n}\right)\right\} \to \phi(u).$$

Therefore, by the Glivenko Theorem, μ is the weak limit of the compound Poisson distributions.■

Remarks

- Glivenko Theorem: If ϕ_n and ϕ are the charact. functions of μ_n and μ then

$$\phi_n(u) \to \phi(u) \text{ for all } u \in \mathbb{R}^d \Longrightarrow \mu_n \xrightarrow{w} \mu \text{ (weak convergence).}$$

Corollary

The set of all infinitely divisible probab. measures on $\mathbb{R}^d$ coincides with the weak closure of the set of all compound Poisson distributions on $\mathbb{R}^d$.

- Proof: Use the theorem and the property: If $\{\mu_n, n \in \mathbb{N}\}$ are inf. divis. and $\mu_n \xrightarrow{w} \mu$ then μ is inf. div.

Stable random variables

- stable distributions is an important subclass of inf. divis. distributions
- Let $d = 1$ and $\{Y_n, n \in \mathbb{N}\}$ be a sequence of iid r.v. We consider the general central limit problem. Define the rescaled partial sums sequence:

$$S_n = \frac{Y_1 + \cdots + Y_n - b_n}{\sigma_n}.$$

- $\{b_n, n \in \mathbb{N}\}$:sequence of real numbers and $\{\sigma_n, n \in \mathbb{N}\}$: sequence of positive numbers.
- Problem: When exists a r.v. X such that

$$\lim_{n\to\infty} P(S_n \leq x) = \lim_{n\to\infty} P(X \leq x) \quad ? \tag{4}$$

In that case S_n converges in distribution to X.
- Usual central limit theorem: if $b_n = nm$, $\sigma_n = \sqrt{n}\sigma$. Then $X \sim N(m, \sigma^2)$.

Stable Random variables

- A r.v. is said to be stable if it arises as a limit as in (4)
- This is equivalent to:

Definition

A r.v. X is said to be stable if exist real valued sequences $\{c_n, n \in \mathbb{N}\}$, $\{d_n, n \in \mathbb{N}\}$ with each $c_n > 0$, such that

$$X_1 + \cdots + X_n \stackrel{d}{=} c_n X + d_n, \qquad (5)$$

where $X_1, \ldots, X_n$ are independent copies of X. In particular, it is strictly stable if each $d_n = 0$.

- In fact, it can be proved that if X is stable then $\sigma_n = \sigma n^{\frac{1}{\alpha}}$ with $0 < \alpha \leq 2$. The parameter α is called the index of stability.
- (5) is equivalent to
$$\phi_X(u)^n = e^{iud_n}\phi_X(c_n u).$$
- All stable random variables are infinitely divisible (trivial consequence of (5)).

Stable Random variables

Theorem

If X is a stable r.v. then:

1. *when $\alpha = 2$, $X \sim N(b, A)$*
2. *when $\alpha \neq 2$, $A = 0$ and*

$$\nu(dx) = \begin{cases} \frac{c_1}{x^{1+\alpha}}dx & \text{if } x > 0 \\ \frac{c_2}{|x|^{1+\alpha}}dx & \text{if } x < 0. \end{cases}, \quad \text{where } c_1, c_2 \geq 0 \text{ and } c_1 + c_2 > 0.$$

Proof can be found in the Book of sato, p. 80.

Stable Random variables

Theorem

A r.v. X is stable if and only if exist $\sigma > 0$, $-1 \leq \beta \leq 1$ *and* $\mu \in \mathbb{R}$ *such that*

1. *when* $\alpha = 2$,
$$\phi_X(u) = \exp\left(i\mu u - \frac{1}{2}\sigma^2 u^2\right);$$
2. *when* $\alpha \neq 1, 2$
$$\phi_X(u) = \exp\left(i\mu u - \sigma^\alpha |u|^\alpha \left[1 - i\beta \operatorname{sgn}(u) \tan\left(\frac{\pi\alpha}{2}\right)\right]\right)$$
3. *when* $\alpha = 1$,
$$\phi_X(u) = \exp\left(i\mu u - \sigma |u| \left[1 + i\beta \frac{2}{\pi} \operatorname{sgn}(u) \log(|u|)\right]\right)$$

Proof can be found in Sato, p. 86.

Stable Random variables

- $E\left[X^2\right] < \infty$ if and only if $\alpha = 2$ (only if X is Gaussian).
- $E\left[|X|\right] < \infty$ if and only if $1 < \alpha \leq 2$.
- All stable r.v. X have densities f_X. In general, can be expressed in series form, but in 3 cases, we have a closed form.
- **Normal distribution:** $\alpha = 2$ and $X \sim N\left(\mu, \sigma^2\right)$.
- **Cauchy distribution:** $\alpha = 1$, $\beta = 0$, $f_X(x) = \frac{\sigma}{\pi\left[(x-\mu)^2 + \sigma^2\right]}$.
- **Lévy distribution**: $\alpha = \frac{1}{2}$, $\beta = 1$,

$$f_X(x) = \left(\frac{\sigma}{2\pi}\right)^{\frac{1}{2}} \frac{1}{(x-\mu)^{\frac{3}{2}}} \exp\left[\frac{\sigma}{-2(x-\mu)}\right] \quad \text{for } x > \mu.$$

Stable Random variables

- Exercise 7: Let X and Y be independent standard normal random variables (with mean 0). Show that Z has a Cauchy distribution, where $Z = X/Y$ if $Y \neq 0$ and $Z = 0$ if $Y = 0$.
- Remark: if X is stable and symmetric then

$$\phi_X(u) = \exp\left(-\rho^\alpha |u|^\alpha\right) \quad \text{for all } 0 < \alpha \leq 2.$$

where $\rho = \sigma$ for $0 < \alpha < 2$ and $\rho = \frac{\sigma}{\sqrt{2}}$ when $\alpha = 2$.

- Important feature of stable laws:when $\alpha \neq 2$ the decay of the tails is polynomial (slow decay $\Longrightarrow$ "heavy tails") -(if $\alpha = 2$ the decay is exponential):

$$P[X > x] \sim \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi x}} \quad \text{if } \alpha = 2,$$

$$\lim_{x\to\infty} x^\alpha P[X > x] \sim C_\alpha \frac{1+\beta}{2}\sigma^\alpha \text{ if } \alpha \neq 2, \text{ with } C_\alpha > 1.$$

Stable Random variables

- All the previous results can be extended to random variablew with values in $\mathbb{R}^d$. Just replace $X_1, \ldots, X_n, X$ and each d_n in (5) by vectors and adapt the previous theorems.
- Note that when $\alpha \neq 2$ and $d > 1$, then the Lévy measure is given by

$$\nu(dx) = \frac{c}{|x|^{d+\alpha}}dx, \quad \text{where } c > 0.$$

- Applebaum, D. (2004). Lévy Processes and Stochastic Caculus. Cambridge University Press. - (Overview and chapter 1)
- Applebaum, D. (2005). Lectures on Lévy Processes, Stochastic Calculus and Financial Applications, Ovronnaz September 2005, Lecture 1 in http://www.applebaum.staff.shef.ac.uk/ovron1.pdf
- Papantaloleon, A. An Introduction to Lévy Processes with Applications in Finance. arXiv:0804.0482v2.
- Sato, K. (1999). Lévy Processes and Infinitely Divisible Distributions. Cambridge University Press.