# Lévy processes and applications - A general introduction

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## Outline



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# Programme

- General introduction and Imperfections of the Black-Scholes model.
- Lévy processes. Definitions, examples and Basic properties
- Stochastic calculus for Lévy processes.
- Stochastic exponentials, exponential martingales and martingale representation theorems
- Lévy processes in finance.

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# Bibliography

Main:

- Schoutens, W (2003), Lévy Processes in Finance, John Wiley & Sons
- Applebaum, D. (2004), Lévy Processes and Stochastic Calculus, Cambridge University Press
- Cont, R. and Tankov, P. (2003), Financial modelling with Jump Processes, Chapman & Hall / CRC Press

Other:

- Bingham, N. H. and Kiesel, R. (2004), Risk-Neutral Valuation: Pricing and Hedging of Financial Derivatives, 2nd. Edition, Springer
- Sato, K.-I. (1999), Lévy Processes and Infinitely Divisible Distributions, Cambridge University Press

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# Assessment

- The final grade, on a 0-20 scale, is awarded on the basis of a final written exam (50%) and a group assignment distributed during the semester (50%)
- Group assignment: the presentation and discussion of an important scientific paper in the field of Lévy processes and applications in finance
- Groups of 3 students

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# Introduction

- **Lévy process**: stochastic process with stationary and independent increments.
- The basic theory was developed by Paul Lévy (1886-1971) on the 1930s.
- Why the interest in Lévy Processes?
- Many interesting examples: Brownian motion, Poisson processes, jump-diffusion processes, subordinated processes, financial models, etc...
- Lévy processes are the simplest generic class of processes with continuous paths interspersed with random jumps at random times.
- Lévy processes are a natural subclass of semimartingales.
- A large class of Markov processes can be built as solutions of stochastic differential equations driven by Lévy noise.

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- Lévy processes have a "robust structure": most applications deal with Lévy processes taking values in Euclidean space but this can be replaced by a Hilbert space or a Banach space (for SPDE's).
- Applications:
  - Turbulence
  - Finance.
  - Quantum Groups
- Main areas in Finance:
  - Option pricing in incomplete markets
  - Interest rate modelling
- Why in Finance?
  - Describe the observed reality in a more accurate way than the usual Brownian motion models: asset prices have jumps; Empirical distribution of the returns exhibits "fat tails" and skewness; the implied volatilities are constant neither across strike nor across maturities.

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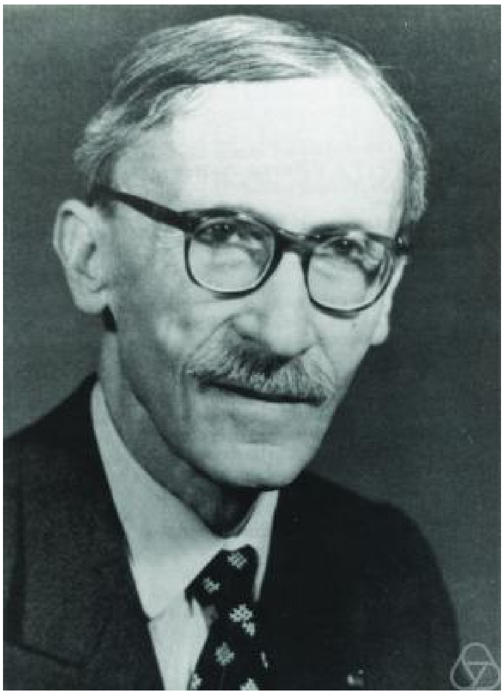
- Paul Lévy (1886-1971)



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- Aleksandr Khintchine (1894-1959)

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# Imperfections of the Black-Scholes model

- Asset price processes have jumps.
- Empirical distribution of asset returns exhibits fat tails and skewness.
- Implied volatilities are constant neither cross strike nor across maturities.

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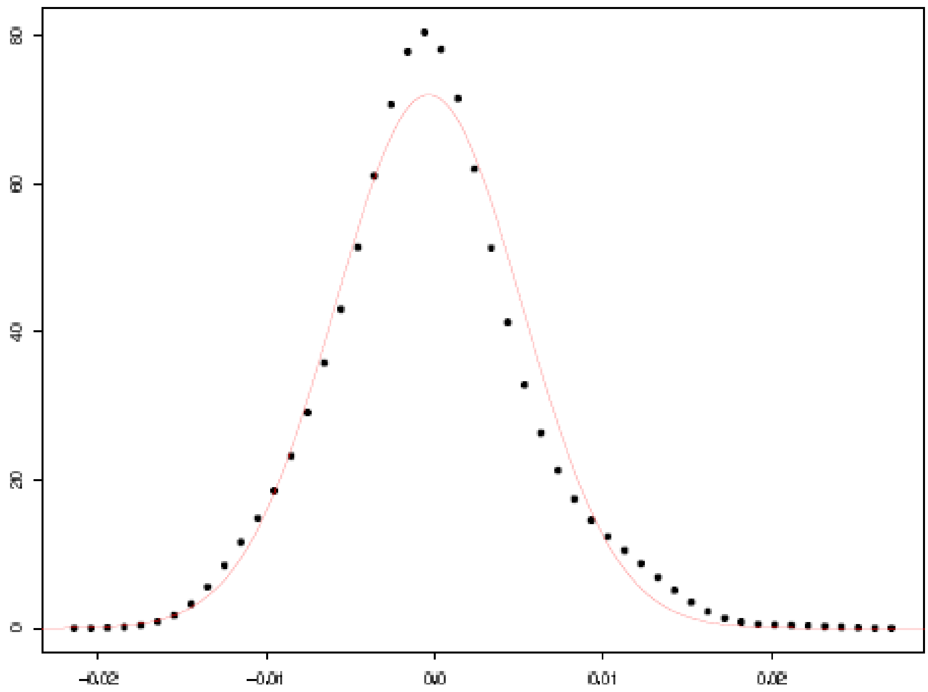


Figure: Empirical Distribution of daily log-returns for the GBP/USD exchange rate and fitted normal distribution

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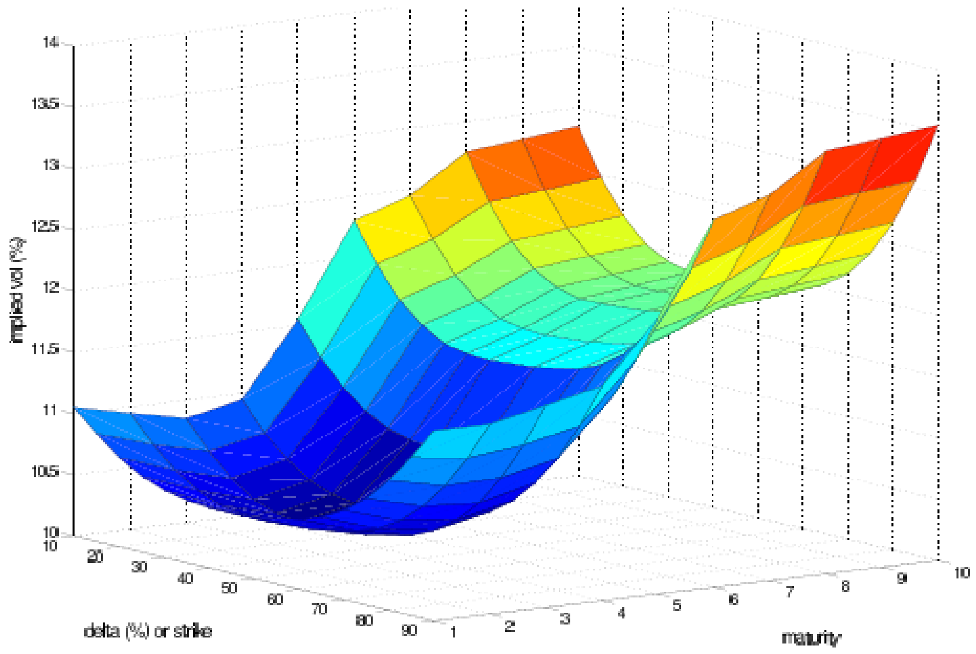


Figure: Implied volatilities of vanilla options on the EUR/USD exchange rate on November 5, 2001.

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# Lévy Processes - Definition

**Definition**

A càdlág, adapted, stochastic process $L = \{L_t, t \in [0,T]\}$ is a **Lévy process** if
$L_0 = 0$ a.s. and
$L$ has independent increments
$L$ has stationary increments
$L$ is stochastically continuous, i.e., for every $t \in [0,T]$ and $\varepsilon > 0$, we have

$$\lim_{s \to t} \mathbb{P}\left[|L_t - L_s| > \varepsilon\right] = 0.$$

- An example (jump-diffusion)

$$L_t = bt + \sigma W_t + \sum_{k=1}^{N_t} J_k - t\lambda m, \qquad (1)$$

where $N$ is a Poisson process with parameter $\lambda$ and $J = (J_k)_{k \geq 1}$ is a i.i.d. sequence with probab. distribution $F$ and $\mathbb{E}[J] = m$.

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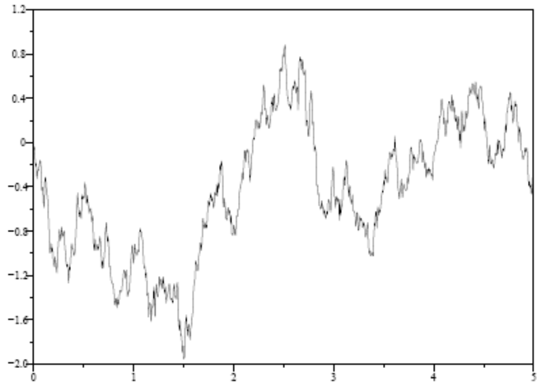

**Figure 1** Simulation of standard Brownian motion

Figure:

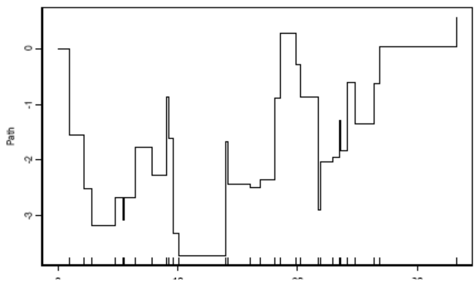

Figure 3. Simulation of a compound Poisson process with $N(0,1)$

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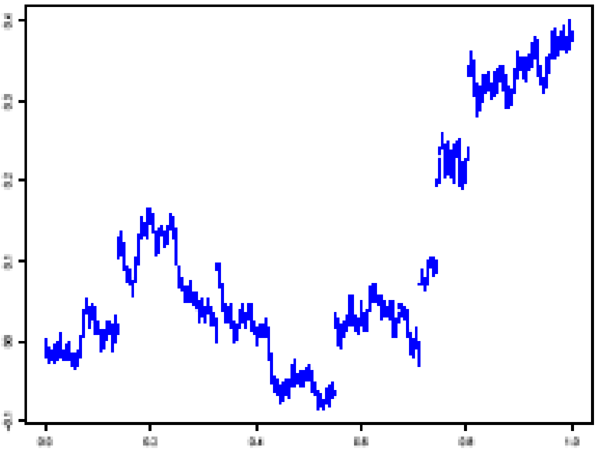

Figure: A jump-diffusion trajectory

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# Infinitely divisible distributions

- The characteristic function of the jump diffusion (1) is

$$\mathbb{E}\left[e^{iuL_t}\right]=\exp\left[t\left(iub-\frac{u^2\sigma^2}{2}+\int_{\mathbb{R}}\left(e^{iux}-1-iux\right)\lambda F\left(dx\right)\right)\right]. \qquad (2)$$

- Sketch of the proof:

$$\mathbb{E}\left[e^{iuL_t}\right]=\exp\left[iubt\right]\mathbb{E}\left[\exp\left[iu\sigma W_t\right]\right]\mathbb{E}\left[\exp\left[iu\sum_{k=1}^{N_t}J_k-iut\lambda m\right]\right].$$

$$\mathbb{E}\left[\exp\left[iu\sigma W_t\right]\right]=\exp\left[-\frac{1}{2}\sigma^2u^2t\right], \qquad W_t\sim N\left(0,t\right),$$

$$\mathbb{E}\left[\exp\left[iu\sum_{k=1}^{N_t}J_k\right]\right]=\exp\left[\lambda t\mathbb{E}\left[e^{iu\,J}-1\right]\right], \qquad N_t\sim Po\left(\lambda t\right).$$

$$\mathbb{E}\left[e^{iuL_t}\right]=\exp\left[iubt-\frac{\sigma^2u^2t}{2}\right]\exp\left[\lambda t\int_{\mathbb{R}}\left(e^{iux}-1-iux\right)\lambda F\left(dx\right)\right].$$

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# Infinitely divisible distributions

Definition

The law $P_X$ of a r.v. $X$ is infinitely divisible if for all $n \in \mathbb{N}$, there exist i.i.d. random variables $X_1^{(1/n)}, X_2^{(1/n)}, \ldots, X_n^{(1/n)}$, such that:

$$X \stackrel{d}{=} X_1^{(1/n)} + X_2^{(1/n)} + \ldots + X_n^{(1/n)}.$$

- $P_X$ is infinitely divisible if, for all $n \in \mathbb{N}$, exists a r.v. $X^{(1/n)}$ such that

$$\varphi_X(u) = \left(\varphi_{X^{(1/n)}}(u)\right)^n.$$

Example

(The Poisson Distribution): $X \sim Po(\lambda)$; $X^{(1/n)} \sim Po\left(\frac{\lambda}{n}\right)$.

$$\begin{aligned}\varphi_X(u) &= \exp\left(\lambda\left(e^{iu} - 1\right)\right) \\ &= \left(\exp\left[\frac{\lambda}{n}\left(e^{iu} - 1\right)\right]\right)^n = \left(\varphi_{X^{(1/n)}}(u)\right)^n.\end{aligned}$$





# Lévy-Kintchine formula

Theorem

*(Lévy Khintchine formula): $P_X$ is infinitely divisible if and only if exists a triplet $(b, c, \nu)$, $b \in \mathbb{R}$, $c \geq 0$, where $\nu$ is a measure, $\nu(\{0\}) = 0$, $\int_{\mathbb{R}} \left(1 \wedge x^2\right) \nu(dx) < \infty$ and*

$$\mathbb{E}\left[e^{iuX}\right] = \exp\left[ibu - \frac{u^2 c}{2} + \int_{\mathbb{R}} \left(e^{iux} - 1 - iux\mathbf{1}_{\{|x|<1\}}\right) \nu(dx)\right].$$

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# Characteristic triplet of a Lévy process

- The triplet $(b, c, \nu)$ is called the Lévy or characteristic triplet and the exponent

$$\psi(u) = ibu - \frac{u^2 c}{2} + \int_{\mathbb{R}} \left(e^{iux} - 1 - iux\mathbf{1}_{\{|x|<1\}}\right) \nu(dx)$$

  is called the Lévy or characteristic exponent.
- $b$ is the drift term, $c$ is the Gaussian or diffusion coefficient and $\nu$ is the Lévy measure.
- The r.v. $L_t$ of the jump diffusion process (1) has infinitely divis. dist. and $b = bt$, $c = \sigma^2 t$ and $\nu = (\lambda F)\, t$.
- Consider a general Lévy process $L = \{L_t, t \in [0, T]\}$. Then

$$L_t = L_{\frac{t}{n}} + \left(L_{\frac{2t}{n}} - L_{\frac{t}{n}}\right) + \cdots + \left(L_t - L_{\frac{(n-1)t}{n}}\right).$$

  By the stationarity and independence of increments, $\left(L_{\frac{kt}{n}} - L_{\frac{(k-1)t}{n}}\right)$ is an iid sequence. Therefore, $L_t$ has an infinitely divisible dist.

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# Lévy-Kintchine formula for a Lévy process

- The characteristic function of a Lévy process is given by the **Lévy-Khintchine formula** (infinitely divisible distribution):

$$\begin{aligned}\phi_u(t) &= \mathbb{E}\left[e^{iuL_t}\right] = \exp\{t\psi(u)\} \\ &= \exp\left\{t\left(ibu - \frac{u^2 c}{2} + \int_{-\infty}^{+\infty} \left(e^{iux} - 1 - iux\mathbf{1}_{\{|x|<1\}}\right) \nu(dx)\right)\right\},\end{aligned}$$

  where $\nu$ is the Lévy measure, $(b, c, \nu)$ is the triplet of characteristics of the Lévy process and $\psi(u)$ is the characteristic exponent of $L_1$.
- Every Lévy process can be associated with a infinitely divisible distribution.
- The opposite (Lévy-Itô decomposition) is also true. Given a r.v. $X$ with infinitely divisible distrib., we can construct a Lévy process $L = \{L_t, t \in [0, T]\}$ such that the law of $L_1$ is the law of $X$.

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# Jumps of a Lévy process

- Jump process: $\Delta L = \{\Delta L_t, t \in [0, T]\}$, where

$$\Delta L_t = L_t - L_{t-}.$$

- By the stochastic continuity of $L$, for a fixed $t$, $\Delta L_t = 0$ a.s.
- It is possible that

$$\sum_{s \leq t} |\Delta L_s| = \infty \quad \text{a.s.}$$

- However,

$$\sum_{s \leq t} |\Delta L_s|^2 < \infty \quad \text{a.s.}$$

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# Poisson random measures

- Let $A \in \mathcal{B}(\mathbb{R} \setminus \{0\})$ such that $0 \in \overline{A}$. The Poisson random measure of the jumps:

$$\mu^L(\omega, t, A) = \#\{0 \leq s \leq t; \Delta L_s \in A\} = \sum_{s \leq t} \mathbf{1}_A(\Delta L_s(\omega)).$$

- $\mu^L(\cdot, A)$ has independent and stationary increments.
- Hence, $\mu^L(\cdot, A)$ is a Poisson process and $\mu^L$ is called a Poisson random measure.
- The measure $\nu$ defined on $\mathcal{B}(\mathbb{R} \setminus \{0\})$ by

$$\nu(A) = \mathbb{E}\left[\mu^L(1, A)\right] = \mathbb{E}\left[\sum_{s \leq 1} \mathbf{1}_A(\Delta L_s(\omega))\right]$$

is the Lévy measure of the Lévy process $L$.

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# Poisson random measures

- Let $f:\mathbb{R}\to\mathbb{R}$ be a bounded measurable function on $A$. Then, the integral of $f$ with respect to Poisson random measure is defined by

$$\int_A f(x)\,\mu^L(\omega,t,dx) = \sum_{s\le t} f(\Delta L_s)\,\mathbf{1}_A(\Delta L_s(\omega)).$$

- Each $\int_A f(x)\,\mu^L(t,dx)$ is a r.v. and $\int_0^t\int_A f(x)\,\mu^L(ds,dx)$ is a stochastic process.

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Theorem

*The process $\int_0^t\int_A f(x)\,\mu^L(ds,dx)$ is a compound Poisson process with characteristic function*

$$\exp\left(t\int_A\left(e^{iuf(x)}-1\right)\nu(dx)\right).$$

*If $f\in L^1(A)$ then*

$$\mathbb{E}\left[\int_0^t\int_A f(x)\,\mu^L(ds,dx)\right] = t\int_A f(x)\,\nu(dx).$$

*If $f\in L^2(A)$ then*

$$\mathrm{Var}\left(\left|\int_0^t\int_A f(x)\,\mu^L(ds,dx)\right|\right) = t\int_A |f(x)|^2\,\nu(dx).$$

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# The Lévy-Itô decomposition

Theorem

*Consider a triple $(b, c, \nu)$ of an inf. divisible law. Then there exists a prob. space and 4 independent Lévy processes $L^{(1)}$, $L^{(2)}$, $L^{(3)}$ and $L^{(4)}$ such that*

$$L = L^{(1)} + L^{(2)} + L^{(3)} + L^{(4)}$$

*is a Lévy process with characteristic triplet $(b, c, \nu)$ and*

$$L_t^{(1)} = bt; \quad L_t^{(2)} = \sqrt{c} W_t,$$

$$L_t^{(3)} = \int_0^t \int_{|x|\geq 1} x \mu^L (ds, dx),$$

$$L_t^{(4)} = \int_0^t \int_{|x|<1} x \left(\mu^L - \nu^L\right)(ds, dx).$$

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# The Lévy measure, paths and moment properties

- $\nu$ satisfies $\nu(\{0\}) = 0$, $\int_{\mathbb{R}} \left(1 \wedge x^2\right) \nu(dx) < \infty$ and describes the expected number of jumps at a certain level in a time interval of size 1.
- If $\nu(\{\mathbb{R}\}) = \infty$ then infinitely many jumps occur (small jumps). The Lévy process has infinite activity.
- If $\nu(\{\mathbb{R}\}) < \infty$ then a.a. paths have a finite number of jumps. The Lévy process has finite activity.
- Let $L$ be a Lévy process with triplet $(b, c, \nu)$. If $c = 0$ and $\int_{|x|\leq 1} |x| \nu(dx) < \infty$ then a.a. paths have finite variation. If $c \neq 0$ or $\int_{|x|\leq 1} |x| \nu(dx) = \infty$ then a.a. paths have infinite variation.

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# The Lévy measure, paths and moment properties

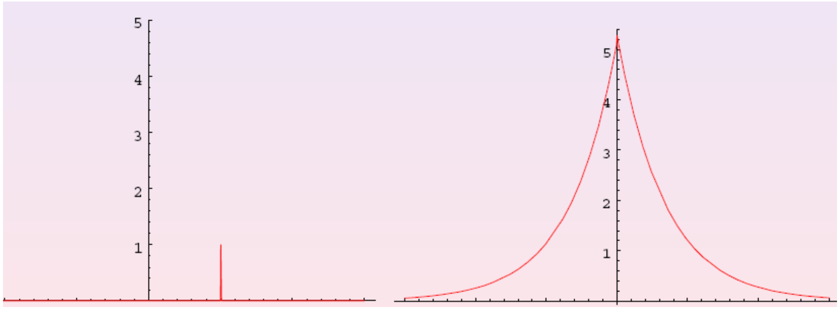


Figure: The Lévy measure of the Poisson and of a compound Poisson process

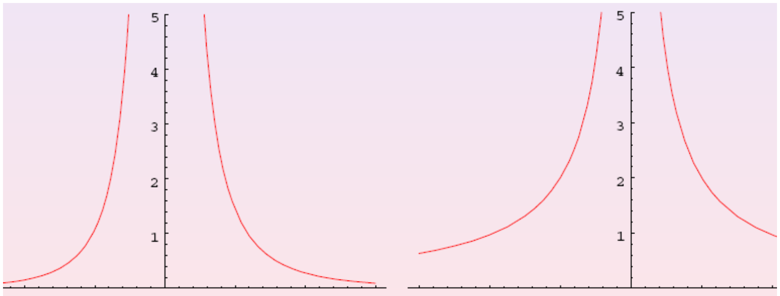


Figure: The Lévy measure of a NIG and an $\alpha$-stable process

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The Lévy-Itô decomposition

# The Lévy measure, paths and moment properties

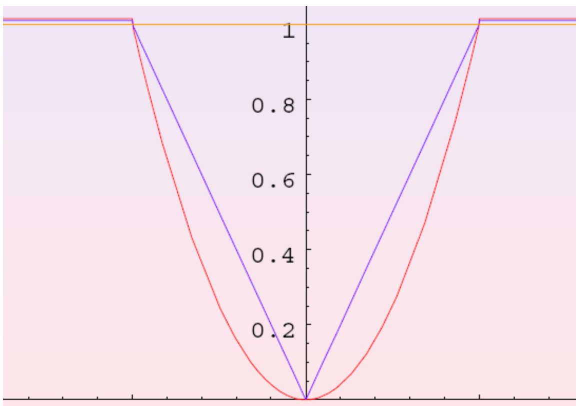


Figure: $|x|^2 \wedge 1$ (red). $|x| \wedge 1$ (blue)

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# The Lévy measure, paths and moment properties

- The path variation properties depend on the small jumps (and Brownian motion).
- The activity depends on all the jumps.
- The moment properties depend on the big jumps.
- The finiteness of the moments of a Lévy processes is related to the finiteness of an integral over the Lévy measure (considering only big jumps).
- $L_t$ has finite moment of order $p$ iff $\int_{|x|\geq 1} |x|^p \nu(dx) < \infty$.
- $L_t$ has finite exponential moment of order $p$ $\left(\text{i.e. } \mathbb{E}\left[e^{pL_t}\right] < \infty\right)$ iff $\int_{|x|\geq 1} e^{px} \nu(dx) < \infty$.

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# Models

- Subordinator: it is an a.s. increasing (in $t$) Lévy process.
- A Lévy process is a subordinator if $\nu(-\infty, 0) = 0$, $c = 0$, $\int_{(0,1)} x\nu(dx) < \infty$ and $b \geq 0$.
- The characteristic exponent is

$$\psi(u) = ibu + \int_0^{\infty} \left(e^{iux} - 1\right) \nu(dx)$$

- The Poisson process is a subordinator.

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# Asset price models

In the risk neutral-world, the asset price process is

$$S_t = S_0 \exp(L_t), \qquad 0 \leq t \leq T$$

- $L_t$ is a Lévy process with triplet $(\overline{b}, \overline{c}, \overline{\nu})$.and canonical decomposition

$$L_t = \overline{b}t + \sqrt{\overline{c}}\overline{W}_t + \int_0^t \int_{\mathbb{R}} x \left(\mu^L - \overline{\nu}^L\right)(ds, dx)$$

with

$$\overline{b} = r - q - \frac{\overline{c}}{2} - \int_{\mathbb{R}} (e^x - 1 - x)\, \overline{\nu}(dx)$$

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# Option pricing

- Transform methods
- PDIE's methods
- Monte-Carlo methods

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# Models

- Black-Scholes model: $L_1 \sim N(\mu, \sigma^2)$. The Lévy triplet is $(\mu, \sigma^2, 0)$ and $L_t = \mu t + \sigma W_t$.
- Merton (jump-diffusion) model: $L_t = \mu t + \sigma W_t + \sum_{k=1}^{N_t} J_k$, with $J_k \sim N(\mu_J, \sigma_J^2)$ (with density $f_J$). The Lévy triplet is $(\mu, \sigma^2, \lambda \times f_J)$.
- Generalized Hyperbolic model: $L_1 \sim GH(\alpha, \beta, \delta, \mu, \lambda)$ and $L_t = t\mathbb{E}[L_1] + \int_0^t \int_{\mathbb{R}} x \left(\mu^L - \nu^{GH}\right)(ds, dx)$. Lévy triplet: $\left(\mathbb{E}[L_1], 0, \nu^{GH}\right)$.
- $\alpha > 0$: related to the shape; $0 \leq |\beta| < \alpha$: skewness.; $\mu$: location; $\delta > 0$ is a scaling parameter; $\lambda$ related with "fat tails".

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# Models

- The Variance Gamma process: It has a characteristic function given by a Variance Gamma distribution $VG(\sigma, \nu, \theta)$ and:

$$\phi_u(t) = \left(1 - iu\theta v + \frac{1}{2}\sigma^2 \nu u^2\right)^{-\frac{t}{\nu}}$$

  It has Lévy triplet $(\gamma, 0, \nu_{VG}(dx))$.
- The Variance Gamma process can be defined as a time-changed Brownian motion with drift:

$$L_t = \theta G_t + \sigma W_{G_t},$$

  where $G$ is a Gamma process with two appropriate parameters.
- Normal inverse Gaussian model (NIG)
- CGMY model
- Meixner model
- etc...

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- Applebaum, D. (2004). Lévy Processes and Stochastic Caculus. Cambridge University Press.
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