Option pricing with Lévy processes

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Stochastic exponential

- Let $d = 1$ and consider the process $Z = (Z(t), t \geq 0)$ solution of the SDE:

$$dZ(t) = Z(t-)\, dY(t), \tag{1}$$

where Y is a Lévy-type stochastic integral.

- The solution is the "stochastic exponential" or "Doléans-Dade exponential":

$$Z(t) = \mathcal{E}_Y(t) = \exp\left\{Y(t) - \frac{1}{2}[Y_c, Y_c](t)\right\} \prod_{0 \leq s \leq t} (1 + \Delta Y(s))\, e^{-\Delta Y(s)}. \tag{2}$$

- We require that (assumption):

$$\inf\{\Delta Y(t), t \geq 0\} > -1 \text{ a.s.} \tag{3}$$

Stochastic exponential

Proposition

If Y is a Lévy-type stochastic integral and (3) holds, then each $\mathcal{E}_Y(t)$ is a.s. finite.

- Exercise: Prove the previous proposition (see Applebaum)
- Note that (3) also implies that $\mathcal{E}_Y(t) > 0$ a.s.
- The stochastic exponential $\mathcal{E}_Y(t)$ is the unique solution of SDE (1) which satisfies the initial condition $Z(0) = 1$ a.s.
- If (3) does not hold then $\mathcal{E}_Y(t)$ may take negative values.

Stochastic exponential

- Alternative form of (2):

$$\mathcal{E}_Y(t) = e^{S_Y(t)}, \tag{4}$$

where

$$\begin{aligned} dS_Y(t) &= F(t)\,dB(t) + \left(G(t) - \frac{1}{2}F(t)^2\right)dt \\ &\quad + \int_{|x|\geq 1} \log(1 + K(t,x))\,N(dt,dx) + \int_{|x|<1} \log(1 + H(t,x))\,\widetilde{N}(dt,dx) \\ &\quad + \int_{|x|<1} (\log(1 + H(t,x)) - H(t.x))\,\nu(dx)\,dt \end{aligned} \tag{5}$$

Stochastic exponential

Theorem

$$d\mathcal{E}_Y(t) = \mathcal{E}_Y(t)\, dY(t)$$

- Exercise: Prove the previous theorem by applying the Itô formula to (5) (see Applebaum).

Stochastic exponential

- Example 1: If $Y(t) = \sigma B(t)$, where $\sigma > 0$ and B is a BM, then

$$\mathcal{E}_Y(t) = \exp\left\{\sigma B(t) - \frac{1}{2}\sigma^2 t\right\}.$$

- Example 2: If $Y = (Y(t), t \geq 0)$ is a compound Poisson process: $Y(t) = X_1 + \cdots + X_{N(t)}$ then

$$\mathcal{E}_Y(t) = \prod_{i=1}^{N(t)} (1 + X_j)$$

Stochastic exponential

- Let X be a Lévy process with characteristics (b, σ, ν) and Lévy-Itô decomposition $X(t) = bt + \sigma B(t) + \int_{|x|<1} x\widetilde{N}(t, dx) + \int_{|x|\geq 1} xN(t, dx)$.
- When can $\mathcal{E}_X(t)$ be written as $\exp(X_1(t))$ for a certain Lévy process X_1 and vice-versa?
- By (4) and (5) we have $\mathcal{E}_X(t) = e^{S_X(t)}$ with

$$S_X(t) = \sigma B(t) + \int_{|x|\geq 1} \log(1+x) N(t, dx) + \int_{|x|<1} \log(1+x) \widetilde{N}(t, dx) + t\left[b - \frac{1}{2}\sigma^2 + \int_{|x|<1} (\log(1+x) - x)\nu(dx)\right]. \quad (6)$$

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Stochastic exponential

- Comparing the Lévy-Itô decomposition with (6), we have

Theorem

If X is a Lévy process with each $\mathcal{E}_X(t)$, then $\mathcal{E}_X(t) = \exp(X_1(t))$ where X_1 is a Lévy process with characteristics (b_1, σ_1, ν_1) given by:

$$\nu_1 = \nu \circ f^{-1}, \quad f(x) = \log(1+x).$$

$$b_1 = b - \frac{1}{2}\sigma^2 + \int_{\mathbb{R}-\{0\}} \left[\log(1+x)\chi_{\widehat{B}}(\log(1+x)) - x\chi_{\widehat{B}}(x)\right]\nu(dx),$$

$$\sigma_1 = \sigma.$$

Conversely, there exists a Lévy process X_2 with characteristics (b_2, σ_2, ν_2) such that $\exp(X(t)) = \mathcal{E}_{X_2}(t)$, where

$$\nu_1 = \nu \circ g^{-1}, \quad g(x) = e^x - 1$$

$$b_2 = b + \frac{1}{2}\sigma^2 + \int_{\mathbb{R}-\{0\}} \left[(e^x - 1)\chi_{\widehat{B}}(e^x - 1) - x\chi_{\widehat{B}}(x)\right]\nu(dx),$$

$$\sigma_2 = \sigma.$$

Exponential martingales

- Lévy-type stochastic integral:

$$dY(t) = G(t)\,dt + F(t)\,dB(t) + \int_{|x|<1} H(t,x)\,\widetilde{N}(dt,dx) + \int_{|x|\geq 1} K(t,x)\,N(dt,dx).$$

- When is Y a martingale?
- Assumptions (stronger than necessary to avoid the local martingale concept):
- (M1) $\mathbb{E}\left[\int_0^t \int_{|x|\geq 1} |K(s,x)|^2 \nu(dx)\,ds\right] < \infty$ for each $t > 0$
- (M2) $\int_0^t \mathbb{E}[|G(s)|]\,ds < \infty$ for each $t > 0$.

Exponential martingales

Exponential martingales

- consequence of (M1) and Cauchy-Schwarz inequality: $\int_0^t \int_{|x|\geq 1} |K(s,x)|\,\nu(dx)\,ds < \infty$ a.s.and

$$\int_0^t \int_{|x|\geq 1} K(s,x)\,N(ds,dx) = \int_0^t \int_{|x|\geq 1} K(s,x)\,\widetilde{N}(ds,dx) + \int_0^t \int_{|x|\geq 1} K(s,x)\,\nu(dx$$

and the compensated integral is a martingale.

Theorem

With assumptions (M1) and (M2), Y is a martingale if and only if

$$G(t) + \int_{|x|\geq 1} K(t,x)\,\nu(dx) = 0 \quad \text{(a.s.) for a.a. } t \geq 0.$$

(see the proof in Applebaum)

Exponential martingales

- Let us consider the process $e^{Y} = \left(e^{Y(t)}, t \geq 0\right)$.
- By Itô's formula, we have that

$$e^{Y(t)} = 1 + \int_0^t e^{Y(s-)} F(s)\, dB(s) + \int_0^t \int_{|x|<1} e^{Y(s-)} \left(e^{H(s,x)} - 1\right) \widetilde{N}(ds, dx)$$
$$+ \int_0^t \int_{|x|\geq 1} e^{Y(s-)} \left(e^{K(s,x)} - 1\right) \widetilde{N}(ds, dx)$$
$$+ \int_0^t e^{Y(s-)} \left(G(s) + \frac{1}{2} F(s)^2 + \int_{|x|<1} \left(e^{H(s,x)} - 1 - H(s,x)\right) \nu(dx) \right.$$
$$\left. + \int_{|x|\geq 1} \left(e^{K(s,x)} - 1\right) \nu(dx) \right) ds \tag{7}$$

Exponential martingales

Exponential martingales

Theorem

e^{Y} is a martingale if and only if

$$G(s) + \frac{1}{2} F(s)^2 + \int_{|x|<1} \left(e^{H(s,x)} - 1 - H(s,x)\right) \nu(dx)$$
$$+ \int_{|x|\geq 1} \left(e^{K(s,x)} - 1\right) \nu(dx) = 0 \tag{8}$$

a.s. and for a.a. $s \geq 0$.

- Therefore, e^{Y} is a martingale if and only if

$$e^{Y(t)} = 1 + \int_0^t e^{Y(s-)} F(s)\, dB(s) + \int_0^t \int_{|x|<1} e^{Y(s-)} \left(e^{H(s,x)} - 1\right) \widetilde{N}(ds, dx)$$
$$+ \int_0^t \int_{|x|\geq 1} e^{Y(s-)} \left(e^{K(s,x)} - 1\right) \widetilde{N}(ds, dx).$$

Exponential martingales

- If e^Y is a martingale then $\mathbb{E}\left[e^{Y(t)}\right] = 1$ for all $t \geq 0$ and e^Y is called an exponential martingale.
- if Y is a Brownian integral: $Y(t) = \int_0^t G(s)\, ds + \int_0^t F(s)\, dB(s)$ then (8) is $G(t) = -\frac{1}{2} F(t)^2$ and

$$e^{Y(t)} = \exp\left(\int_0^t F(s)\, dB(s) - \frac{1}{2}\int_0^t F(s)^2\, ds\right).$$

Change of Measure - Girsanov's Theorem

- Let P and Q be two different probability measures. Q_t and P_t are the measures restricted to $(\Omega, \mathcal{F}_t)$.
- Let e^Y be an exponential martingale and define Q_t by

$$\frac{dQ_t}{dP_t} = e^{Y(t)}.$$

- Fix an interval $[0, T]$ and define $P = P_T$ and $Q = Q_T$.

Lemma

$M = (M(t), 0 \leq t \leq T)$ is a Q-martingale if and only if $Me^Y = (M(t)e^{Y(t)}, 0 \leq t \leq T)$ is a P-martingale.

Change of Measure - Girsanov's Theorem

- Let Y be a Brownian integral and $e^{Y(t)} = \exp\left(\int_0^t F(s)\,dB(s) - \frac{1}{2}\int_0^t F(s)^2\,ds\right)$.
- Define a new process

$$B_Q(t) = B(t) - \int_0^t F(s)\,ds.$$

Theorem

(Girsanov): B_Q is a Q-Brownian motion.

- Generalization of Girsanov: Let M be a martingale of the form $M(t) = \int_0^t \int_A L(x,s)\,\widetilde{N}(ds,dx)$, with L predictable, $L \in \mathcal{P}_2$. Then

$$N(t) = M(t) - \int_0^t \int_A L(s,x)\left(e^{H(s,x)} - 1\right)\nu(dx)\,ds$$

is a Q-martingale.

Option pricing

- Stock price: $S = (S(t), t \geq 0)$.
- Contingent claims with maturity date T: Z is a non-negative $\mathcal{F}_T$ measurable r.v. representing the payoff of the option.
- European call option: $Z = \max\{S(T) - K, 0\}$
- American call option: $Z = \sup_{0 \leq \tau \leq T} \max\{S(\tau) - K, 0\}$
- Asian option: $Z = \max\left\{\frac{1}{T}\int_0^T (S(t) - K)\,dt, 0\right\}$
- We assume that the interest rate r is constant.
- Discounted stock price process: $\widetilde{S} = \left(\widetilde{S}(t), t \geq 0\right)$ with $\widetilde{S}(t) = e^{-rt}S(t)$.
- Portfolio: $(\alpha(t), \beta(t))$, $\alpha(t)$ is the number of stocks and $\beta(t)$ the number of riskless assets (bonds).
- Portfolio value: $V(t) = \alpha(t)S(t) + \beta(t)A(t)$
- A portfolio is said to be replicating if $V(T) = Z$.

Option pricing

- Self-financing portfolio: $dV(t) = \alpha(t)\, dS(t) + r\beta(t)\, A(t)\, dt$.
- A market is said to be complete if every contingent claim can be replicated by a self-financing portfolio.
- An arbitrage opportunity exists if the market allows risk-free profit. The market is arbitrage free if there exists no self-financing strategy for which $V(0) = 0$, $V(T) \geq 0$ and $P(V(T) > 0) > 0$.

Theorem

(Fundamental Theorem of Asset Pricing 1 in discrete time) If the market is free of arbitrage opportunities, then there exists a probability measure Q, which is equivalent to P, with respect to which the discounted process $\widetilde{S}$ is a martingale.

- A similar result holds in the continuous case but we need to make more technical assumptions - instead of absence of arbitrage we need the stronger NFLVR hypothesis ("no free lunch with vanishing risk").

Option pricing

Theorem

Fundamental Theorem of Asset Pricing 2) An arbitrage-free market is complete if and only if there exists a unique probability measure Q, which is equivalent to P, with respect to which the discounted process $\widetilde{S}$ is a martingale.

- Such a Q is called a martingale measure or risk-neutral measure.
- If Q exists, but is not unique, then the market is said to be incomplete.
- In a complete market, it turns out that we have

$$V(t) = e^{-r(T-t)}\mathbb{E}_Q\left[Z|\mathcal{F}_t\right]$$

and this is the arbitrage-free price of the claim Z at time t.

Stock price as a Lévy process

- Return:

$$\frac{\delta S(t)}{S(t)} = \sigma \delta X(t) + \mu \delta t,$$

where $X = (X(t), t \geq 0)$ is a semimartingale and $\sigma > 0, \mu$ are parameters called the volatility and stock drift.

- Itô calculus SDE:

$$\begin{aligned} dS(t) &= \sigma S(t-) dX(t) + \mu S(t-) dt \\ &= S(t-) dZ(t), \end{aligned}$$

where $Z(t) = \sigma X(t) + \mu t$.

- Then $S(t) = \mathcal{E}_{Z(t)}$ is the stochastic exponential of the semimartingale Z.

Stock price as a Lévy process

- When X is a standard Brownian motion B, we obtain geometric Brownian motion

$$S(t) = \exp\left(\sigma B(t) + \left(\mu - \frac{1}{2}\sigma^2\right) t\right)$$

- idea: Let X be a Lévy process. In order for stock prices to be non-negative, (3) yields $\Delta X(t) > -\sigma^{-1}$ (a.s.) for each t > 0. Denote $c = -\sigma^{-1}$.
- We impose $\int_{(c,-1]\cup[1,+\infty)} x^2 \nu(dx) < \infty$. This means that each $X(t)$ has first and second moments (reasonable for stock returns).
- By the Lévy-Itô decomposition,

$$X(t) = mt + kB(t) + \int_c^\infty x \widetilde{N}(t, dx),$$

where $k \geq 0$ and $m = b + \int_{(c,-1]\cup[1,+\infty)} x\nu(dx)$ (in terms of the earlier parameters).

Stock price as a Lévy process

- Representing $S(t)$ as the stochastic exponential $\mathcal{E}_{Z(t)}$, we obtain from (5) that

$$d\left(\log\left(S\left(t\right)\right)\right) = k\sigma dB(t) + \left(m\sigma + \mu - \frac{1}{2}k^2\sigma^2\right)dt$$
$$+ \int_c^\infty \log\left(1+\sigma x\right)\widetilde{N}\left(dt, dx\right) + \int_c^\infty \left(\log\left(1+\sigma x\right) - \sigma x\right)\nu\left(dx\right)dt$$

- There are a number of explicit mathematically tractable and realistic models: variance-gamma, normal inverse Gaussian, hyperbolic, etc.

Change of measure

- we seek to find measures Q, which are equivalent to P, with respect to which the discounted stock process $\widetilde{S}$ is a martingale.
- Let Y be a Lévy-type stochastic integral of the form:

$$dY(t) = G\left(t\right)dt + F\left(t\right)dB\left(t\right) + \int_{\mathbb{R}-\{0\}} H\left(t,x\right)\widetilde{N}\left(dt,dx\right).$$

- Consider that e^Y is an exponential martingale (therefore, G is determined by F and H).
- Define Q by $\frac{dQ}{dP} = e^{Y(T)}$. By Girsanov theorem and its generalization:

$$B_Q\left(t\right) = B\left(t\right) - \int_0^t F\left(s\right)ds \text{ is a } Q\text{-BM}$$
$$\widetilde{N}_Q(t,A) = \widetilde{N}(t,A) - \nu_Q\left(t,A\right) \text{ is a } Q\text{-martingale}$$
$$\nu_Q\left(t,A\right) := \int_0^t \int_A \left(e^{H(s,x)} - 1\right)\nu\left(dx\right)ds.$$

Change of measure

- $\widetilde{S}(t) = e^{-rt}S(t)$ can be written in terms of these processes by:

$$d\left(\log\left(\widetilde{S}(t)\right)\right) = k\sigma dB_Q(t) + \left(m\sigma + \mu - r - \frac{1}{2}k^2\sigma^2 + k\sigma F(t)\right.$$
$$\left. + \sigma\int_{\mathbb{R}-\{0\}} x\left(e^{H(t,x)} - 1\right)\nu(dx)\right)dt + \int_c^\infty \log(1+\sigma x)\widetilde{N}_Q(dt,dx)$$
$$+ \int_c^\infty (\log(1+\sigma x) - \sigma x)\,\nu_Q(dt,dx).$$

- Put $\widetilde{S}(t) = \widetilde{S}_1(t)\widetilde{S}_2(t)$, where

$$d\left(\log\left(\widetilde{S}_1(t)\right)\right) = k\sigma dB_Q(t) - \frac{1}{2}k^2\sigma^2 dt$$
$$+ \int_c^\infty \log(1+\sigma x)\widetilde{N}_Q(dt,dx) + \int_c^\infty (\log(1+\sigma x) - \sigma x)\,\nu_Q(dt,dx).$$

Change of measure

- and

$$d\left(\log\left(\widetilde{S}_2(t)\right)\right) = \left(m\sigma + \mu - r + k\sigma F(t) + \right.$$
$$\left. + \sigma\int_{\mathbb{R}-\{0\}} x\left(e^{H(t,x)} - 1\right)\nu(dx)\right)dt.$$

- Apllying Itô's formula to $\widetilde{S}_1$ we obtain:

$$d\widetilde{S}_1(t) = k\sigma\widetilde{S}_1(t-)\,dB_Q(t) + \int_c^\infty \sigma\widetilde{S}_1(t-)\,x\widetilde{N}_Q(dt,dx).$$

  and $\widetilde{S}_1$ is a Q-martingale.
- Therefore $\widetilde{S}$ is a Q-martingale if and only if

$$m\sigma + \mu - r + k\sigma F(t) + \sigma\int_{\mathbb{R}-\{0\}} x\left(e^{H(t,x)} - 1\right)\nu(dx) = 0 \text{ a.s.} \qquad (9)$$

Change of measure

- Equation (9) clearly has an infinite number of possible solution pairs (F, H).
- There are an infinite number of possible measures Q with respect to which $\widetilde{S}$ is a martingale. So the general Lévy process model gives rise to incomplete markets.
- Example: the Brownian motion case: $\nu = 0$ and $k \neq 0$. Then there is a unique solution

$$F(t) = \frac{r - \mu - m\sigma}{k\sigma} \quad \text{a.s.}$$

  and the market is complete (Black-Scholes model).

Incomplete markets and Esscher transform

- Equivalent measures Q exist with respect to which $\widetilde{S}$ will be a martingale, but these will no longer be unique in general
- We must follow a selection principle to reduce the class of all possible measures Q to a subclass, within which a unique measure can be found.
- Aditional assumption:

$$\int_{|x|\geq 1} e^{ux} \nu(dx) < \infty$$

  for all $u \in \mathbb{R}$.
- In this case, we can analytically continue the Lévy- Khintchine formula to obtain

$$\mathbb{E}\left[e^{-uX(t)}\right] = e^{-t\psi(u)}$$

  where

$$\psi(u) = -\eta(iu) = bu - \frac{1}{2}k^2u^2 + \int_c^\infty \left(1 - e^{-u\,y} - u\,y\chi_{\widehat{B}}(y)\right)\nu(dy).$$

Incomplete markets and Esscher transform

- The processes

$$M_u(t) = \exp(iuX(t) - t\eta(u)),$$
$$N_u(t) = M_{iu}(t) = \exp(-uX(t) + t\psi(u))$$

are martingales and N_u is strictly positive.

- Define a new probability measure by

$$\frac{dQ_u}{dP}|_{\mathcal{F}_t} = N_u(t).$$

- Q_u is called the Esscher transform of P by N_u.

Incomplete markets and Esscher transform

- Applying Itô formula to N_u, we have

$$dN_u(t) = N_u(t-)\left(-kuB(t) + \left(e^{-ux} - 1\right)\widetilde{N}(dt, dx)\right).$$

- Comparing this with (7) for exponential martingales e^Y, we have that

$$F(t) = -ku,$$
$$H(t, x) = -ux$$

and the condition for Q_u to be a martingale (9) is

$$m\sigma + \mu - r - k^2 u\sigma + \sigma\int_c^\infty x\left(e^{-ux} - 1\right)\nu(dx) = 0 \text{ a.s.}$$

Incomplete markets and Esscher transform

- Let $z(u) = \int_c^{\infty} x\left(e^{-ux} - 1\right)\nu(dx) - k^2 u$. Then the martingale condition is:

$$z(u) = \frac{r - \mu - m\sigma}{\sigma}. \tag{10}$$

- Since $z'(u) < 0$, z is strictly decrerasing, and therefore there is a unique u (a unique measure Q_u) that satisfies (10).

Hyperbolic processes in finance

- Let $A \in \mathcal{B}(\mathbb{R})$ be measurable set and let $(g_\theta, \theta \in A)$ be a family of probability density functions, and ρ a probability distribution on A (called mixing measure).
- The "probability mixture"

$$h(x) = \int_A g_\theta(x)\,\rho(d\theta)$$

  is a probability density function on $\mathbb{R}$.
- The hyperbolic distributions are "probability mixtures".
- Bessel functions of the 3rd kind:

$$K_\nu(x) = \frac{1}{2}\int_0^{\infty} u^{\nu-1}\exp\left(-\frac{1}{2}x\left(u + \frac{1}{u}\right)\right)du, \quad x, \nu \in \mathbb{R}.$$

- For each $a, b > 0$

$$f_\nu^{a,b}(x) = \frac{\left(\frac{a}{b}\right)^{\frac{\nu}{2}}}{2K_\nu\left(\sqrt{ab}\right)} x^{\nu-1}\exp\left(-\frac{1}{2}\left(ax + \frac{b}{x}\right)\right)$$

  is a pdf on $(0, \infty)$ - called a Generalized Inverse Gaussian or $GIG(\nu, a, b)$.

Hyperbolic processes in finance

- Take ρ to be $GIG(1, a, b)$ and $A = (0, \infty)$ and g_{σ^2} the pdf of $N(\mu + b\sigma^2, \sigma^2)$ with $\mu, b \in \mathbb{R}$.
- The resulting probability mixture is

$$h_{\delta,u}^{\alpha,\beta}(x) = \frac{\sqrt{\alpha^2 - \beta^2}}{2\alpha\delta K_1\left(\delta\sqrt{\alpha^2 - \beta^2}\right)} \exp\left(-\alpha\sqrt{\delta^2 + (x-\mu)^2} + \beta(x-\mu)\right),$$

  for all $x \in \mathbb{R}$, where $\alpha^2 = a + \beta^2$ and $\delta^2 = b$.
- The corresponding law is called an hyperbolic distribution (*log* $\left(h_{\delta,u}^{\alpha,\beta}\right)$ is a hyperbola). Parameters: μ (location), α (tail), β (asymmetry), and δ (scale).
- These dist. are infinitely divisible and all their moments exist.

Hyperbolic processes in finance

Hyperbolic processes in finance

- Moment generating function: $M_{\delta,u}^{\alpha,\beta}(u) = \int_{\mathbb{R}} e^{ux} h_{\delta,u}^{\alpha,\beta}(x)\, dx$
- It can be proved that

$$M_{\delta,u}^{\alpha,\beta}(u) = e^{\mu u} \frac{\sqrt{\alpha^2 - \beta^2}}{K_1\left(\delta\sqrt{\alpha^2 - \beta^2}\right)} \frac{K_1\left(\delta\sqrt{\alpha^2 - (\beta + u^2)}\right)}{\sqrt{\alpha^2 - (\beta + u^2)}}$$

- The characteristic function is $\phi(u) = M(iu)$
- For simplicity, we restrict to the symmetric case ($\mu = \beta = 0$) and with $\zeta = \delta\alpha$,

$$h_{\zeta,\delta}(x) = \frac{1}{2\delta K_1(\zeta)} \exp\left(-\zeta\sqrt{1 + \left(\frac{x}{\delta}\right)^2}\right).$$

  The corresponding Lévy process has no Gaussian part and it is:

$$X_{\zeta,\delta}(t) = \int_0^t \int_{\mathbb{R}-\{0\}} x\widetilde{N}(ds, dx).$$

Option pricing with hyperbolic processes

- Stock price:
$$dS(t) = S(t-)\,dX_{\zeta,\delta}(t).$$
- A drawback of this approach is that the jumps of $X_{\zeta,\delta}$ are not bounded below (they can be < -1). That is why we model:
$$S(t) = S(0)\,e^{X_{\zeta,\delta}(t)},$$
$$\widetilde{S}(t) = S(0)\,e^{X_{\zeta,\delta}(t)-rt}.$$
- Martingale measure Q such that $\widetilde{S}$ is a Q martingale. Since the market is incomplete, we can use the Esscher transform and
$$\frac{dQ_u}{dP}|_{\mathcal{F}_t} = N_u(t) = \exp\left(-uX_{\zeta,\delta}(t) - t\log\left(M_{\zeta,\delta}(u)\right)\right).$$

Hyperbolic processes in finance

Option pricing with hyperbolic processes

- By the Generalized Girsanov theorem, $\widetilde{S}$ is a Q-martingale iff $\widetilde{S}N_u$ is a P-martingale.
- $$\widetilde{S}(t)\,N_u(t) = \exp\left((1-u)\,X_{\zeta,\delta}(t) - t\left(\log\left(M_{\zeta,\delta}(u)\right) + r\right)\right).$$
- On the other hand, it can be proved that
$$\exp\left((1-u)\,X_{\zeta,\delta}(t) - t\log\left(M_{\zeta,\delta}(1-u)\right)\right)$$
is a martingale.
- Therefore $\widetilde{S}$ is a Q-martingale iff
$$r = \log\left(M_{\zeta,\delta}(1-u)\right) - \log\left(M_{\zeta,\delta}(1-u)\right) =$$
$$= \log\left[\frac{K_1\sqrt{\zeta^2 - \delta^2(1-u)^2}}{K_1\left(\sqrt{\zeta^2-\delta^2u^2}\right)}\right] - \frac{1}{2}\log\left[\frac{\zeta^2-\delta^2(1-u)^2}{\zeta^2-\delta^2u^2}\right].$$

Option pricing with hyperbolic processes

- The value of u can be determined from the previous expression by numerical means.
- We can now price an European call option by

$$V(0) = \mathbb{E}_{Q_u}\left[e\left(se^{X_{\zeta,\delta}(T)} - K\right)^+\right]$$

- If $f_{\zeta,\delta}^{(t)}$ is the pdf of $X_{\zeta,\delta}(t)$ with respect to P then the Esscher transform can be used to show that $X_{\zeta,\delta}(t)$ has the pdf with respect to Q_u

$$f_{\zeta,\delta}^{(t)}(x;u) = f_{\zeta,\delta}^{(t)}(x)\, e^{-ux - t\log(M_{\zeta,\delta}(u))}.$$

- Then, the pricing formula is:

$$V(0) = s\int_{\log\left(\frac{k}{x}\right)}^{\infty} f_{\zeta,\delta}^{(T)}(x;1-u)\,dx - e^{-rT}K\int_{\log\left(\frac{k}{x}\right)}^{\infty} f_{\zeta,\delta}^{(T)}(x;u)\,dx.$$

Option pricing with hyperbolic processes

- Volatility: If we had $S(t) = e^{Z(t)}$ with $Z(t) = \sigma B(t)$ (where B is a B.M.) then the volatility is $\sigma^2 = \mathbb{E}\left[Z(1)^2\right]$.
- By analogy, in the hyperbolic case the volatility is defined by $\sigma^2 = \mathbb{E}\left[X_{\zeta,\delta}(1)^2\right]$ and can be proved that (from the moment generating function and Bessel functions properties):

$$\sigma^2 = \frac{\delta^2 K_2(\zeta)}{\zeta K_1(\zeta)}.$$

- Applebaum, D. (2004). Lévy Processes and Stochastic Caculus. Cambridge University Press.
- Applebaum, D. (2005). Lectures on Lévy Processes, Stochastic Calculus and Financial Applications, Ovronnaz September 2005, in http://www.applebaum.staff.shef.ac.uk/ovron3.pdf
- Cont, R. and Tankov, P. (2003). Financial modelling with jump processes. Chapman and Hall/CRC Press.
- Papantaloleon, A. An Introduction to Lévy Processes with Applications in Finance. arXiv:0804.0482v2.
- Sato, K. (1999). Lévy Processes and Infinitely Divisible Distributions. Cambridge University Press.
- Schoutens, W. (2002). Lévy Processes in Finance: Pricing Financial Derivatives. Wiley.