Lévy processes and applications - Theoretical overview

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Lévy-Khintchine formula

- $(\Omega, \mathcal{F}, \mathbf{P}).$
- A Lévy Process X = (X(t), t ≥ 0) is essentially a stochastic process with stationary and indepedent increments.
- Key-formula: the Lévy-Khintchine formula:

$$\mathsf{E}\left[\mathsf{e}^{i(u,X(t))}\right] = \mathsf{e}^{t\eta(u)}$$

where

$$\eta(u) = i(b, u) - \frac{1}{2}(u, Au) + \int_{\mathbb{R}^d - \{0\}} \left[e^{i(u, y)} - 1 - i(u, y) \chi_{(0 < |y| < 1)}(y) \right] \nu(dy),$$

 $b \in \mathbb{R}^d$, A is a $d \times d$ positive definite symmetric matrix and a ν is a Lévy measure on $\mathbb{R}^d - \{0\}$ such that, for all $u \in \mathbb{R}^d$:

$$\int_{\mathbb{R}^{d}-\{0\}}\left(\left|\boldsymbol{y}\right|^{2}\wedge\mathbf{1}\right)\nu\left(\boldsymbol{d}\boldsymbol{y}\right)<\infty.$$

• Let $A = \nu = 0$. Then

$$E\left[e^{i(u,X(t))}
ight]=e^{it(u,b)}$$

and X(t) = bt is a deterministic motion in a stright line (*b* is the velocity of the motion - drift)

• Let $A \neq 0$ and $\nu = 0$. Then

$$E\left[e^{i(u,X(t))}\right] = \exp\left[t\left[i\left(b,u\right) - \frac{1}{2}\left(u,Au\right)\right]\right],$$

which is the characteristic of a Gaussian r.v. with mean *tb* and covariance matrix *tA*.In fact X(t) is a Brownian motion with drift.

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Lévy-Khintchine formula

• Let ν be a finite measure ($\lambda = \int_{\mathbb{R}^d} \nu(d\mathbf{x}) < \infty$). Then

$$\eta(u) = i(b', u) - \frac{1}{2}(u, Au) + \int_{\mathbb{R}^d - \{0\}} \left[e^{i(u, y)} - 1\right] \nu(dy),$$

with $b' = b - \int_{0 < |y| < 1} y\nu(dy)$. • If $\nu = \lambda \delta_h$ with $\lambda > 0$ then

$$X(t) = b't + \sigma B(t) + N(t),$$

where $\sigma = \sqrt[n]{A}$ (or $\sigma \sigma^T = A$) and N(t) is a Poisson process of intensity λ with jumps of size |h|. Note that $E\left[e^{i(u,N(t))}\right] = \exp\left[\lambda t\left(e^{i(u,h)}-1\right)\right]$

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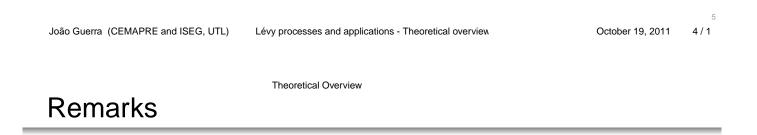
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• Let $\nu = \sum_{i=1}^{m} \lambda_i \delta_{h_i}$. Then

$$X(t) = b't + \sigma B(t) + N_1(t) + \cdots N_m(t),$$

where the N'_i s are independent Poisson processes (also independent of *B*), with N_i of intensity λ_i with jumps of size $|h_i|$.



- Most subtle case: infinite measure case: $\int_{0 < |y| < 1} |y| \nu(dy) = \infty$ and $\int_{0 < |y| < 1} |y|^2 \nu(dy) < \infty$.
- Then $e^{i(u,y)} 1$ may no longer be ν -integrable, but $e^{i(u,y)} 1 i(u,y) \chi_{(0 < |y| < 1)}(y)$ is allways ν -integrable
- intuition: ν has become so fine that an infinite number of small jumps is expected.
- When ν is finite we can write:

$$X(t) = bt + \sigma B(t) + \sum_{0 \le s \le t} \Delta X(s),$$

• $\Delta X(s)$ is the jump at time *s*.

- For each Borel set $A \in \mathbb{R}^d \{0\}$, let $N(t, A) = \# \{0 \le s \le t : \Delta X(s) \in A\}$
- Fix t and A: then N(t, A) is a r.v.
- Fix $\omega \in \Omega$ and *t*: then $N(t, \cdot)(\omega)$ is a measure
- Fix A. Then $\{N(t, A), t \ge 0\}$ is a Poisson process with intensity $\nu(A)$ with

$$\sum_{0\leq s\leq t}\Delta X\left(s\right)=\int_{\mathbb{R}-\left\{0\right\}}xN\left(t,dx\right)$$

• In the case of infinite measure ν , we have the Lévy-Itô decomposition:

$$X(t) = bt + \sigma B(t) + \int_{0 < |x| < 1} x [N(t, dx) - t\nu(dx)] + \int_{|x| \ge 1} x N(t, dx)$$

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Theoretical Overview

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- Papantaloleon, A. An Introduction to Lévy Processes with Applications in Finance. arXiv:0804.0482v2.