Lévy-Itô decomposition and stochastic integration

João Guerra

CEMAPRE and ISEG, UTL

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Processes of Finite Variation

- Let $\mathcal{P} = \{a = t_1 < t_2 < \cdots < t_n < t_{n+1} = b\}$ be a partition of $[a, b] \subset \mathbb{R}$, with diameter $\delta = \max_{1 \leq i \leq n} |t_{i+1} - t_i|$.
- Variation $Var_{\mathcal{P}}[g]$ of a càdlàg function g over partition $\mathcal{P}$:

$$Var_{\mathcal{P}}[g] := \sum_{i=1}^{n} |g(t_{i+1}) - g(t_i)| .$$

- If $V[g] := \sup_{\mathcal{P}} Var_{\mathcal{P}}[g] < \infty$, we say g has finite variation on $[a, b]$.
- If g is defined on $\mathbb{R}$ (or $\mathbb{R}^+$), we say it has finite variation if it has finite variation on each compact interval.
- Every non-decreasing g has finite variation.

Processes of Finite Variation

- Functions of finite variation are important in integration:if we propose g as an integrator, in order to define the Stieltjes integral: $\int_I fdg$ for all continuous functions f, a necessary and sufficient condition for obtaining $\int_I fdg$ as a limit of Riemann sums is that g has finite variation.
- A stochastic process $(X(t), t \geq 0)$ is of finite variation if the paths $(X(t)(\omega), t \geq 0)$ are of finite variation for almost all $\omega \in \Omega$.

Example - Poisson integrals

- N: Poisson random measure with intensity measure μ, let f be a measurable function and A bounded below. Let

$$Y(t) = \int_A f(x) N(t, dx).$$

- The process Y has finite variation on $[0, t]$ for each $t \geq 0$.
- Indeed:

$$Var_{\mathcal{P}}[Y] \leq \sum_{0 \leq s \leq t} |f(\Delta X(s))| \chi_A(\Delta X(s)) < \infty \quad \text{a.s. },$$

  where $X(t) = \int_A xN(t, dx)$ for each $t \geq 0$.
- Necessary and sufficient condition for a Lévy process to be of finite variation: there is no Brownian part ($A = 0$ in the Lévy-Khinchine formula), and

$$\int_{|x|<1} |x| \nu(dx) < \infty.$$

Lévy-Itô decomposition

- For A bounded below,

$$\int_A xN(t,dx) = \sum_{0\le s\le t} \Delta X(s)\,\chi_A(\Delta X(s)).$$

  is the sum of all the jumps taking values in A, up to time t.
- paths of X are càdlàg $\Longrightarrow$ the sum is a finite random sum.In particular, $\int_{|x|\ge 1} xN(t,dx)$ is finite ("big jumps"). It is a compound Poisson process, has finite variation but may have no finite moments.
- Conversely, $X(t) - \int_{|x|\ge 1} xN(t,dx)$ is a Lévy process with finite moments of all orders.
- Exercise: show that if X is a Lévy process with bounded jumps then we have $E(|X(t)|^m) < \infty$ for all $m \in \mathbb{N}$. (Hint: see proof of theorem 2.4.7, pages 118-119 of Applebaum).

Lévy-Itô decomposition

- For small jumps, let us consider compensated integrals (which are martingales): (A bounded below)

$$M(t,A) := \int_A x\widetilde{N}(t,dx).$$

- Consider the "ring-sets":

$$B_m := \left\{ x \in \mathbb{R}^d : \frac{1}{m+1} < |x| \le \frac{1}{m} \right\},$$

$$A_n := \bigcup_{m=1}^{n} B_m.$$

- We can show that

$$\int_{|x|<1} x\widetilde{N}(t,dx) = L^2 - \lim_{n\to\infty} M(t,A_n).$$

  Therefore $\int_{|x|<1} x\widetilde{N}(t,dx)$ is a martingale (the L^2 limit of a sequence of martingales) .

Lévy-Itô decomposition

- Taking the limit in $E\left[\exp\left\{i\left(u,\int_{A_n} x\widetilde{N}(t,dx)\right)\right\}\right] = \exp\left(t\int_{\mathbb{R}^d}\left(e^{i(u,x)}-1-i(u,x)\right)\mu_{X,A_n}(dx)\right)$ (see Poisson integration in the previous session), we obtain

$$E\left[\exp\left\{i\left(u,\int_{|x|<1} x\widetilde{N}(t,dx)\right)\right\}\right] = \exp\left(t\int_{|x|<1}\left(e^{i(u,x)}-1-i(u,x)\right)\mu(dx)\right)$$

- Consider

$$B_A(t) = X(t) - bt - \int_{|x|<1} x\widetilde{N}(t,dx) - \int_{|x|\geq 1} xN(t,dx),$$

where $b = \mathbb{E}\left(X(1) - \int_{|x|\geq 1} xN(1,dx)\right)$.

- B_A is a centered martingale with continuous paths and has covariance matrix A.
- By the Lévy characterization of B.M., B_A is a Brownian motion with covariance matrix A.

Lévy-Itô decomposition

Theorem

(Lévy-Itô decomposition): If X is a Lévy process, then exists $b\in\mathbb{R}^d$, a Brownian motion B_A with covariance matrix A and an independent Poisson random measure N on $\mathbb{R}^+\times\left(\mathbb{R}^d-\{0\}\right)$ such that

$$X(t) = bt + B_A(t) + \int_{|x|<1} x\widetilde{N}(t,dx) + \int_{|x|\geq 1} xN(t,dx). \quad (1)$$

- The 3 processes in (1) are independent.

Lévy-Itô decomposition

- The Lévy-Khintchine formula is a corollary of the Lévy-Itô decomposition.

Corollary

(Lévy-Khintchine formula): If X is a Lévy process then

$$E\left[e^{i(u,X(t))}\right]=\exp\left\{t\left[i(b,u)-\frac{1}{2}(u,Au)+\int_{\mathbb{R}^d-\{0\}}\left[e^{i(u,y)}-1-i(u,y)\chi_B(y)\right]\mu(dy)\right]\right\}$$

- The intensity measure μ is the Lévy measure for X.
- $\int_{|x|<1} x\widetilde{N}(t,dx)$ is the compensated sum of small jumps (it is an L^2-martingale).
- $\int_{|x|\geq 1} xN(t,dx)$ is the sum of large jumps (compound Poisson process, but may have no finite moments).

Lévy-Itô decomposition

- A Lévy process has finite variation if its Lévy-Itô decomposition is

$$\begin{aligned} X(t) &= \gamma t+\int_{x\neq 0} xN(t,dx) \\ &= \gamma t+\sum_{0\leq s\leq t}\Delta X(s), \end{aligned}$$

where $\gamma=b-\int_{|x|<1} x\nu(dx)$.

Lévy-Itô decomposition

Financial interpretation for the jump terms in the Lévy-Itô decomposition:

- if intensity measure (μ or ν) is infinite: the stock price has "infinite activity" $\approx$ flutuations and jumpy movements arising from the interaction of pure supply shocks and pure demand shocks.
- if the intensity measure (μ or ν) is finite, we have "finite activity" $\approx$ sudden shocks that can cause unexpected movements in the market, such as a major earthquake.
- If a pure jump Lévy process (no Brownian part) has finite activity $\Longrightarrow$ then it has finite variation. The converse is false.
- The first 3 terms on the rhs of (1) have finite moments to all orders $\Longrightarrow$ if a Lévy process fails to have a moment, this is due to the "large jumps"/"finite activity" part $\int_{|x|\geq 1} xN(t, dx)$.
- $E\left[|X(t)|^n\right] < \infty$ if and only if $\int_{|x|\geq 1} |x|^n \nu(dx) < \infty$.

Semimartingales

Definition

A stochastic process $X = \{X(t), t \geq 0\}$ is a semimartingale if it an adapted process which admits a decomposition:

$$X = X(0) + M(t) + C(t), \tag{2}$$

where M is a local martingale and C is an adapted process of finite variation.

- Semimartingales are "good integrators": largest class of processes with respect to which the Itô integral can be defined.
- A Lévy process is a semimartingale:by (1),

$$M(t) = B_A(t) + \int_{|x|<1} x\widetilde{N}(t, dx),$$

$$C(t) = bt + \int_{|x|\geq 1} xN(t, dx).$$

Stochastic integration

- Let $X = M + C$ be a semimartingale.
- Stochastic integral w.r.t. X:

$$\int_0^t F(s)\, dX_s = \int_0^t F(s)\, dM_s + \int_0^t F(s)\, dC_s. \tag{3}$$

- $\int_0^t F(s)\, dC_s$ defined by the usual Lebesgue-Stieltjes integral.
- In general, $\int_0^t F(s)\, dM_s$ requires a stochastic definition (in general, M has infinite variation).
- Define a "martingale valued measure"

$$M(t, E) = B_t \delta_0(E) + \widetilde{N}(t, E - \{0\}), \tag{4}$$

where B_t is a one-dim. BM and $E \subset \mathbb{R}^n$ is measurable.
- $M((s,t], E) = M(t,E) - M(s,E)$ is independent of $\mathcal{F}_s$.

$$\int_0^t \int_E F(s,x)\, M(ds, dx) = \int_0^t G(s)\, dB_s + \int_0^t \int_{E-\{0\}} F(s,x)\, \widetilde{N}(ds, dx), \tag{5}$$

where $G(s) = F(s, 0)$.

Stochastic integration

- $\mathbb{E}[M((s,t], E)] = 0$,
- $\mathbb{E}\left[(M((s,t], E))^2\right] = \rho((s,t], E)$, where $\rho((s,t], E) := (t-s)(\delta_0(E) + \nu(E - \{0\}))$.

- Let $\mathcal{P}$ be the smallest σ-algebra with respect to which all the mappings $F : [0, T] \times E \times \Omega \to \mathbb{R}$ satisfying (1) and (2) below are measurable:
  1. For each t, $(x, \omega) \to F(t, x, \omega)$ is $\mathcal{B}(E) \times \mathcal{F}_t$ measurable.
  2. For each x and ω, $t \to F(t, x, \omega)$ is left continuous.
- $\mathcal{P}$ is called the predictable σ-algebra. A $\mathcal{P}$-measurable mapping is said predictable.
- Let $\mathcal{H}_2$ be the linear space of mappings $F : [0, T] \times E \times \Omega \to \mathbb{R}$ which are predictable and

$$\int_0^T \int_E \mathbb{E}\left[|F(t,x)|^2\right] \nu(dx)\, dt < \infty, \tag{6}$$

$$\int_0^T \mathbb{E}\left[|F(t,0)|^2\right] \nu dt < \infty. \tag{7}$$

- Let F be a simple process:

$$F = \sum_{j=1}^{m} \sum_{k=1}^{n} F_k(t_j)\, \mathbf{1}_{(t_j, t_{j+1}]} \mathbf{1}_{A_k} \tag{8}$$

- F is predictable and its stochastic integral is defined by

$$I(F) = \sum_{j=1}^{m} \sum_{k=1}^{n} F_k(t_j)\, M\left((t_j, t_{j+1}], A_k\right) \tag{9}$$

Lemma

If F is simple then

$$\mathbb{E}[I(F)] = 0,$$

$$E\left[(I(F))^2\right] = \int_0^T \int_E \mathbb{E}\left[|F(t,x)|^2\right] \nu(dx)\, dt \tag{10}$$

- Exercise: Show that $\mathbb{E}[I(F)] = 0$.
- I is a linear isometry from S (set of simple processes) into $L^2(\Omega)$ and since S is dense in $\mathcal{H}_2$, I can be extended to $\mathcal{H}_2$ and it is a isometry of $\mathcal{H}_2$ into $L^2(\Omega)$.
- For $F \in \mathcal{H}_2$ we define

$$I_t(F) = \int_0^t \int_E F(t,x) M(ds,dx)$$

and

$$\int_0^t \int_E F(t,x) M(ds,dx) = \lim_{n\to\infty} (L^2) \int_0^t \int_E F_n(t,x) M(ds,dx), \qquad (11)$$

where $\{F_n, n \in \mathbb{N}\}$ is a sequence of simple processes.

- The stochastic integral $I_t(F)$ with $F \in \mathcal{H}_2$ satisfies:
  1. I_t is a linear operator
  2. $\mathbb{E}[I(F)] = 0$, $E\left[(I(F))^2\right] = \int_0^T \int_E \mathbb{E}\left[|F(t,x)|^2\right] \nu(dx)\, dt$.
  3. $\{I_t(F),\ t \in [0,T]\}$ is $\{\mathcal{F}_t\}$ adapted
  4. $\{I_t(F),\ t \in [0,T]\}$ is a square-integrable martingale.

Poisson stochastic integrals

- The integral of a predictable process $K(t,x)$ with respect to the compound Poisson process $P_t = \int_A xN(t,dx)$ is defined by

$$\int_0^T \int_A K(t,x) N(dt,dx) = \sum_{0 \le s \le T} K(s, \Delta P_s) \mathbf{1}_A (\Delta P_s). \tag{12}$$

- We can also define

$$\int_0^T \int_A H(t,x) \widetilde{N}(dt,dx) = \int_0^T \int_A H(t,x) N(dt,dx) - \int_0^T \int_A H(t,x) \nu(dx) dt \tag{13}$$

if H is predictable and satisfies (6).

Lévy type stochastic integrals

- We say Y is a Lévy type stochastic integral if

$$Y_t = Y_0 + \int_0^t G(s) ds + \int_0^t F(s) dB_s + \int_0^t \int_{|x|<1} H(s,x) \widetilde{N}(ds,dx) + \int_0^t \int_{|x|>1} K(s,x) N(ds,dx), \tag{14}$$

where we assume that the processes G, F, H and K are predictable and satisfy appropriate integrability conditions.
- Y is a semimartingale.
- Let L be a Lévy process with Lévy triplet (b, c, ν) and let X be a predictable left-continuous process satisfying (6). Then we can construct a Lévy stochastic integral Y_t by

$$dY_t = X_t dL_t.$$

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