Even when Theorem 9.7 is applicable it may not be the simplest way to determine the nature of a stationary point. For example, when $f(x, y) = e^{1/g(x,y)}$, where $g(x, y) = x^2 + 2 + \cos^2 y - 2\cos y$, the test is applicable, but the computations are lengthy. In this case we may express $g(x, y)$ as a sum of squares by writing $g(x, y) = 1 + x^2 + (1 - \cos y)^2$. We see at once that f has relative maxima at the points at which $x^2 = 0$ and $(1 - \cos y)^2 = 0$. These are the points $(0, 2n\pi)$, when n is any integer.

9.13 Exercises

In Exercises 1 through 15, locate and classify the stationary points (if any) of the surfaces having the Cartesian equations given.

1. $z = x^2 + (y - 1)^2$.
2. $z = x^2 - (y - 1)^2$.
3. $z = 1 + x^2 - y^2$.
4. $z = (x - y + 1)^2$.
5. $z = 2x^2 - xy - 3y^2 - 3x + 7y$.
6. $z = x^2 - xy + y^2 - 2x + y$.
7. $z = x^3 - 3xy^2 + y^3$.
8. $z = x^2y^3(6 - x - y)$.
9. $z = x^3 + y^3 - 3xy$.
10. $z = \sin x \cosh y$.
11. $z = e^{2x+3y}(8x^2 - 6xy + 3y^2)$.
12. $z = (5x + 7y - 25)e^{-(x^2+xy+y^2)}$.
13. $z = \sin x \sin y \sin (x + y)$, $\quad 0 \le x \le \pi, 0 \le y \le \pi$.
14. $z = x - 2y + \log \sqrt{x^2 + y^2} + 3 \arctan \dfrac{y}{x}$, $\quad x > 0$.
15. $z = (x^2 + y^2)e^{-(x^2+y^2)}$.
16. Let $f(x, y) = 3x^4 - 4x^2y + y^2$. Show that on every line $y = mx$ the function has a minimum at $(0, 0)$, but that there is no relative minimum in any two-dimensional neighborhood of the origin. Make a sketch indicating the set of points (x, y) at which $f(x, y) > 0$ and the set at which $f(x, y) < 0$.
17. Let $f(x, y) = (3 - x)(3 - y)(x + y - 3)$.
    (a) Make a sketch indicating the set of points (x, y) at which $f(x, y) \ge 0$.
    (b) Find all points (x, y) in the plane at which $D_1 f(x, y) = D_2 f(x, y) = 0$. [*Hint:* $D_1 f(x, y)$ has $(3 - y)$ as a factor.]
    (c) Which of the stationary points are relative maxima? Which are relative minima? Which are neither? Give reasons for your answers.
    (d) Does f have an absolute minimum or an absolute maximum on the whole plane? Give reasons for your answers.
18. Determine all the relative and absolute extreme values and the saddle points for the function $f(x, y) = xy(1 - x^2 - y^2)$ on the square $0 \le x \le 1, 0 \le y \le 1$.
19. Determine constants a and b such that the integral
$$\int_0^1 \{ax + b - f(x)\}^2 \, dx$$
will be as small as possible if (a) $f(x) = x^2$; (b) $f(x) = (x^2 + 1)^{-1}$.
20. Let $f(x, y) = Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F$, where $A > 0$ and $B^2 < AC$.
    (a) Prove that a point (x_1, y_1) exists at which f has a minimum. [*Hint:* Transform the quadratic part to a sum of squares.]
    (b) Prove that $f(x_1, y_1) = Dx_1 + Ey_1 + F$ at this minimum.
    (c) Show that
$$f(x_1, y_1) = \frac{1}{AC - B^2} \begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix}.$$

21. *Method of least squares.* Given n distinct numbers $x_1, \ldots, x_n$ and n further numbers $y_1, \ldots, y_n$ (not necessarily distinct), it is generally impossible to find a straight line $f(x) = ax + b$ which passes through all the points (x_i, y_i), that is, such that $f(x_i) = y_i$ for each i. However, we can try a linear function which makes the "total square error"

$$E(a, b) = \sum_{i=1}^{n} [f(x_i) - y_i]^2$$

a minimum. Determine values of a and b which do this.

22. Extend the method of least squares to 3-space. That is, find a linear function $f(x, y) = ax + by + c$ which minimizes the total square error

$$E(a, b, c) = \sum_{i=1}^{n} [f(x_i, y_i) - z_i]^2,$$

where (x_i, y_i) are n given distinct points and $z_1, \ldots, z_n$ are n given real numbers.

23. Let $z_1, \ldots, z_n$ be n distinct points in m-space. If $\boldsymbol{x} \in \mathbf{R}^m$, define

$$f(\boldsymbol{x}) = \sum_{k=1}^{n} \|\boldsymbol{x} - z_k\|^2 .$$

Prove that f has a minimum at the point $\boldsymbol{a} = \frac{1}{n}\sum_{k=1}^{n} z_k$ (the centroid).

24. Let $\boldsymbol{a}$ be a stationary point of a scalar field f with continuous second-order partial derivatives in an n-ball $B(\boldsymbol{a})$. Prove that f has a saddle point at $\boldsymbol{a}$ if at least two of the diagonal entries of the Hessian matrix $H(\boldsymbol{a})$ have opposite signs.

25. Verify that the scalar field $f(x, y, z) = x^4 + y^4 + z^4 - 4xyz$ has a stationary point at $(1, 1, 1)$, and determine the nature of this stationary point by computing the eigenvalues of its Hessian matrix.

9.14 Extrema with constraints. Lagrange's multipliers

We begin this section with two examples of extremum problems with constraints.

EXAMPLE 1. Given a surface S not passing through the origin, determine those points of S which are nearest to the origin.

EXAMPLE 2. If $f(x, y, z)$ denotes the temperature at (x, y, z), determine the maximum and minimum values of the temperature on a given curve C in 3-space.

Both these examples are special cases of the following general problem: *Determine the extreme values of a scalar field $f(\boldsymbol{x})$ when $\boldsymbol{x}$ is restricted to lie in a given subset of the domain of f.*

In Example 1 the scalar field to be minimized is the distance function,

$$f(x, y, z) = (x^2 + y^2 + z^2)^{1/2};$$