Lévy Processes simulation

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Simulation techniques

- Lévy processes allow more realistic models of price dynamics and than traditional diffusion-based models.
- The price to pay for this is an increased complexity of computations.
- Lévy processes are quite tractable when compared to nonlinear diffusion models, but analytical pricing methods are only available for European options.
- For pricing exotic options, risk management, etc., we need numerical methods.
- Two possible choices of numerical methods: (1) Monte Carlo methods; (2) Numerical solution of partial integro-differential equations (PIDE methods) by finite differences or finite elements schemes.
- For higher dimensions, the computational complexity of PIDE methods grows exponentially with dimension.
- On the contrary, the complexity of Monte Carlo methods grows only linearly with the dimension.
- In higher dimensions there is no alternative to simulation methods $\Longrightarrow$ important to develop efficient algorithms for simulating Levy processes.

Simulation of Standard Brownian motion

- The standard Brownian motion $B = \{B_t, t \geq 0\}$ has normally distributed independent increments $\Longrightarrow$ we discretize time by taking time steps of small size Δt and simulate the values of B at time points $\{n\Delta t, n = 0, 1, ...\}$.
- Then

$$\begin{aligned} B_0 &= 0, \\ B_{n\Delta t} &= B_{(n-1)\Delta t} + \sqrt{\Delta t} z_n, \quad n \geq 1, \end{aligned} \tag{1}$$

where $\{z_n, n = 1, 2, ...\}$ is a series of standard normal random numbers.

Simulation of standard Brownian motion

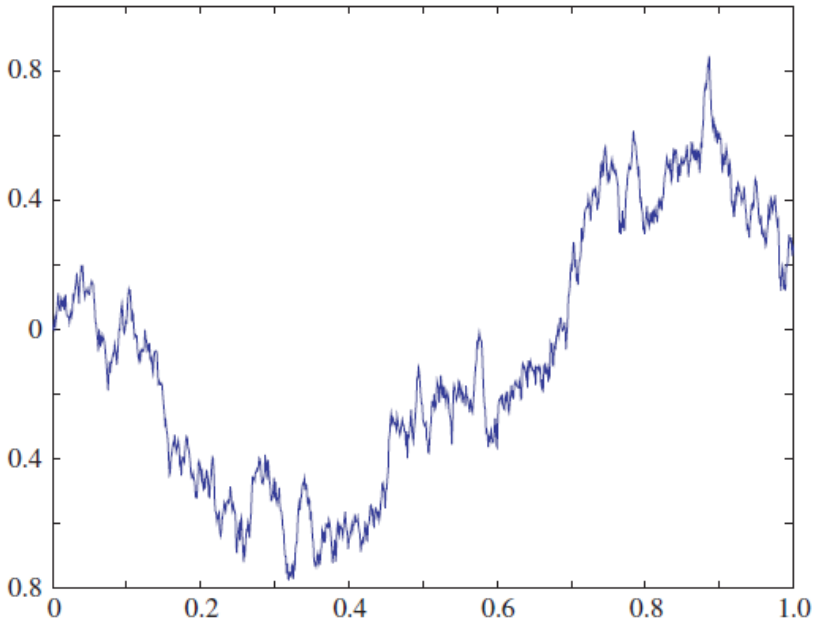

Figure 3.1 A sample path of a standard Brownian motion.

Simulation of a Poisson Process

- The simulation of a Poisson process $N = \{N_t, t \geq 0\}$ with intensity parameter λ can be done in several different ways. We consider two methods.
- (1) The Method of Exponential Spacings:makes use of the fact that the inter-arrival times of the jumps of N follow an Exponential $Exp(\lambda)$ distribution with mean λ^{-1}.
- An $Exp(\lambda)$ random number, e_n, can be obtained from a $Uniform(0,1)$ random number, u_n, by

$$e_n = -\frac{\log(u_n)}{\lambda}.$$

- Let

$$\begin{aligned} s_0 &= 0, \\ s_n &= s_{n-1} + e_n, \quad n = 1, 2, ... \end{aligned} \tag{2}$$

- Simulation of a path of a N in time points $\{n\Delta t, n = 0, 1, ...\}$:

$$\begin{aligned} N_0 &= 0, \\ N_{n\Delta t} &= \sup_k \{k : s_k \leq n\Delta t\}. \end{aligned} \tag{3}$$

Simulation of a Poisson Process

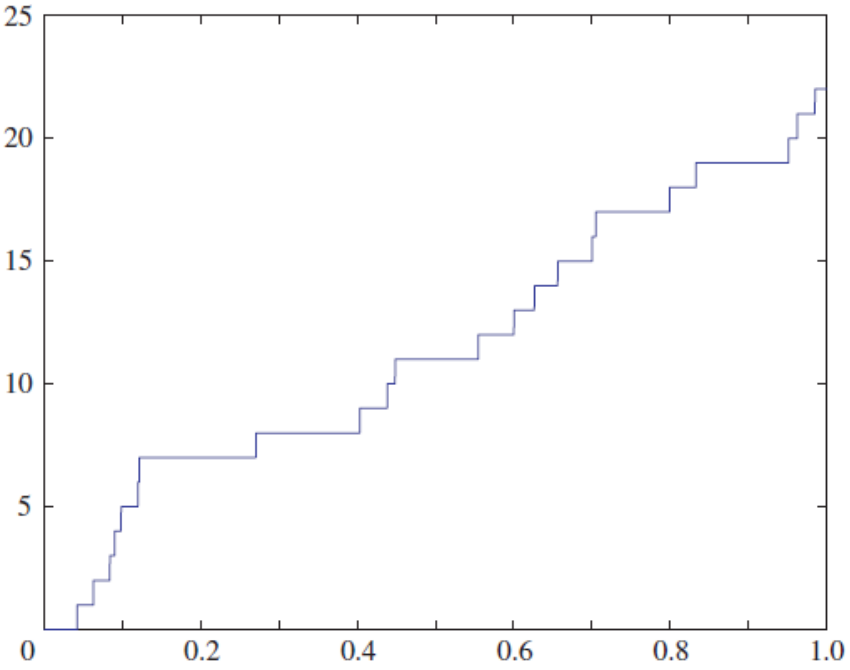

Figure 8.1 A sample path of a Poisson process.

Simulation of a Poisson Process

- (2) The Uniform method: if we need to simulate a Poisson process with intensity parameter $\lambda > 0$ up to a time point $T > 0$,
- (i) first generate a random variate N with distribution $Poisson(\lambda T)$ distributed.
- (ii) simulate N independent random uniform numbers $u_1, .., u_N$. Denote by $u_{(1)} < u_{(2)} < \cdots < u_{(N)}$ the order-statistics of this sequence.
- (iii) Then the jump points of the Poisson process are given by the points $Tu_{(1)}, ..., Tu_{(N)}$, i.e. the Poisson process has a value 0 for time points $t < Tu_{(1)}$. At $t = Tu_{(1)}$ the process jumps to 1 and stays there until $t = Tu_{(2)}$, where it jumps to 2, etc.

Simulation of a Lévy Process

- In order to simulate a Lévy process, we can approximate the process by a compound Poisson process.
- Special care has to be taken for the very small jumps.We can simply replace these very small jumps by their expected values. In some cases a further improvement can be made by replacing these small jumps by a Brownian motion.
- The Compound Poisson approximation for a Lévy process with characteristic triplet $\left[\gamma, \sigma^2, \nu(dx)\right]$:
- (i) discretize the Lévy measure $\nu(dx)$: choose small $0 < \varepsilon < 1$ and make a partition of $\mathbb{R} \setminus [-\varepsilon, \varepsilon]$ .of the following form:

$$a_0 < a_1 < \cdots < a_k = -\varepsilon,\ \varepsilon = a_{k+1} < a_{k+2} < \cdots < a_{d+1}.$$

- (ii) Jumps larger than ε are approximated by a sum of independent Poisson processes: we take an independent Poisson process $N^{(i)}$ for each interval $[a_{i-1}, a_i)$, $i = 1, 2, ..., k$; $[a_i, a_{i+1})$, $i = k+1, ..., d$., with intensity λ_i given by the Lévy measure of the interval i.
- (iii) We choose a point c_i (the jump size) in each interval i such that the variance of $N^{(i)}$ matches the part of the variance of the Lévy Process corresponding to this interval.

Approximation of the small jumps by their expected value

- Approximation of the small jumps by their expected value
- Approximate the original Lévy process $X = \{X_t, t \geq 0\}$ by a process $X^d = \{X_t^d, t \geq 0\}$ with a Brownian motion component and d independent Poisson Processes $N^{(i)}$, with intensity parameter λ_i ($i = 1, 2, ..., d$) (the small jumps ($|c_i| < 1$) are compensated).

$$X_t^d = \gamma t + \sigma B_t + \sum_{i=1}^{d} c_i \left(N_t^{(i)} - \lambda_i t \mathbf{1}_{|c_i|<1} \right), \tag{4}$$

$$\lambda_i = \begin{cases} \nu\left([a_{i-1}, a_i)\right) & \text{for } 1 \leq i \leq k \\ \nu\left([a_i, a_{i+1})\right) & \text{for } k+1 \leq i \leq d \end{cases},$$

$$c_i^2 \lambda_i = \begin{cases} \int_{a_{i-1}}^{a_i} x^2 \nu(dx) & \text{for } 1 \leq i \leq k \\ \int_{a_i}^{a_{i+1}} x^2 \nu(dx) & \text{for } k+1 \leq i \leq d \end{cases}.$$

- If the original process has no Brownian component ($\sigma = 0$), then neither does the approximating process.

Approximation of the small jumps by a Brownian motion

- Improvement - incorporate the contribution from the variation of small jumps:

$$\sigma^2(\varepsilon) = \int_{|x|<\varepsilon} x^2 \nu(dx). \tag{5}$$

- We let all (compensated) jumps smaller than ε contribute to the Brownian part of X: approximate X by a process

$$X_t^d = \gamma t + \widetilde{\sigma} B_t + \sum_{i=1}^{d} c_i \left(N_t^{(i)} - \lambda_i t \mathbf{1}_{|c_i|<1} \right), \tag{6}$$

$$\widetilde{\sigma}^2 = \sigma^2 + \sigma^2(\varepsilon). \tag{7}$$

- A Brownian term now appears even when the original process does not have one ($\sigma = 0$).
- One can show that this type of approximation is valid if

$$\lim_{\varepsilon \to 0} \frac{\sigma(\varepsilon)}{\varepsilon} = \infty. \tag{8}$$

Approximation of the small jumps by a Brownian motion

Special cases:

- The NIG (normal inverse Gaussian) process: replacing the small jumps by a Brownian motion is valid since one can show that $\sigma(\varepsilon) \sim \sqrt{2\alpha\delta/\pi}\varepsilon^{\frac{1}{2}}$ and (8) holds.
- The Meixner Process: in this case one can also show that $\sigma(\varepsilon) \sim \sqrt{2\alpha\delta/\pi}\varepsilon^{\frac{1}{2}}$ and (8) holds and the approximation is also valid.
- The CGMY process: one cans show that if $Y > 0$ then (8) holds. So, one can approximate the small jumps by a Brownian motion if $Y > 0$.
- The Gamma process: In this case $\sigma(\varepsilon)/\varepsilon \to \sqrt{a/2}$ when $\varepsilon \to 0$ and we cannot approximate the small jumps by a Brownian motion.
- The VG (Variance-Gamma) process: since a VG process is the difference of two Gamma processes, we cannot replace the small jumps by a Brownian component in the approximation.

Choice of the Poisson processes

- The choice of the intervals $[a_{i-1}, a_i)$, for $i = 1, 2, ..., k$; $[a_i, a_{i+1})$, for $i = k+1, ..., d$, is crucial.
- For Lévy processes with Lévy measure living on $\mathbb{R}$, we set $d = 2k$, with the same number of Poisson processes for negative jumps and positive jumps.
- 3 ways of choosing the intervals:
- (i) equally space intervals: $|a_i - a_{i-1}|$ is fixed for all i: it has the disadvantage of causing an "explosion" of the values of λ_i for c_i close to zero.
- (ii) equally weighted intervals: we keep the Lévy measures on the intervals fixed, i.e., $\nu([a_{i-1}, a_i))$ is fixed for all $1 \leq i \leq k$ and $\nu([a_i, a_{i+1}))$ is fixed for all $k+1 \leq i \leq d$.
  For this choise, the outher intervals can be quite large.

Choice of the Poisson processes

- (iii) intervals with inverse linear boundaries: the boundaries are given by $a_{i-1} = -\alpha/i$ and $a_{2k+2-i} = \alpha/i$, $1 \leq i \leq k+1$ and $\alpha > 0$.
- The intervals with inverse linear boundaries leads to much more gradually decaying intensity parameters λ_i and there is no explosion to infinity near zero.
- Note also that in all cases the intensities of down-jumps are slightly higher than those of the corresponding up-jumps; this reflects the fact that log returns of stocks are negatively skewed.

Choice of the Poisson processes

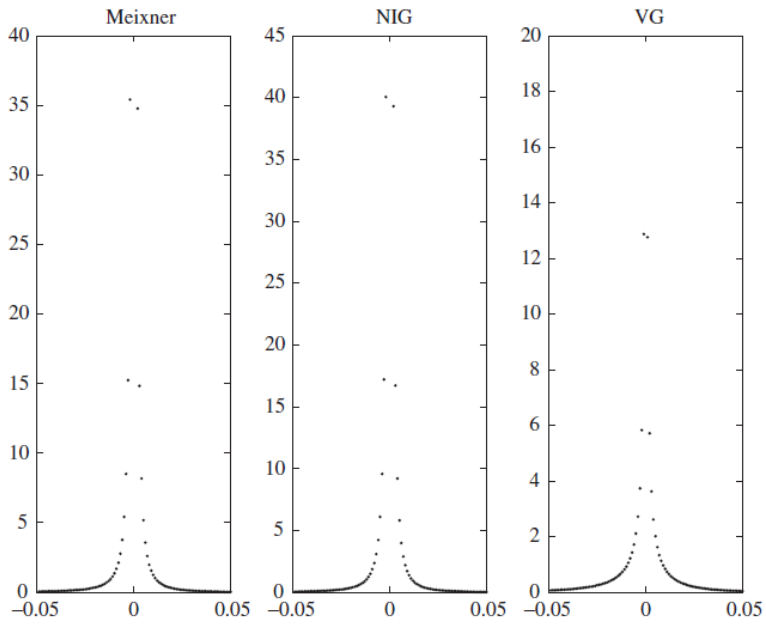

Figure 8.2 Equally spaced intervals.

Choice of the Poisson processes

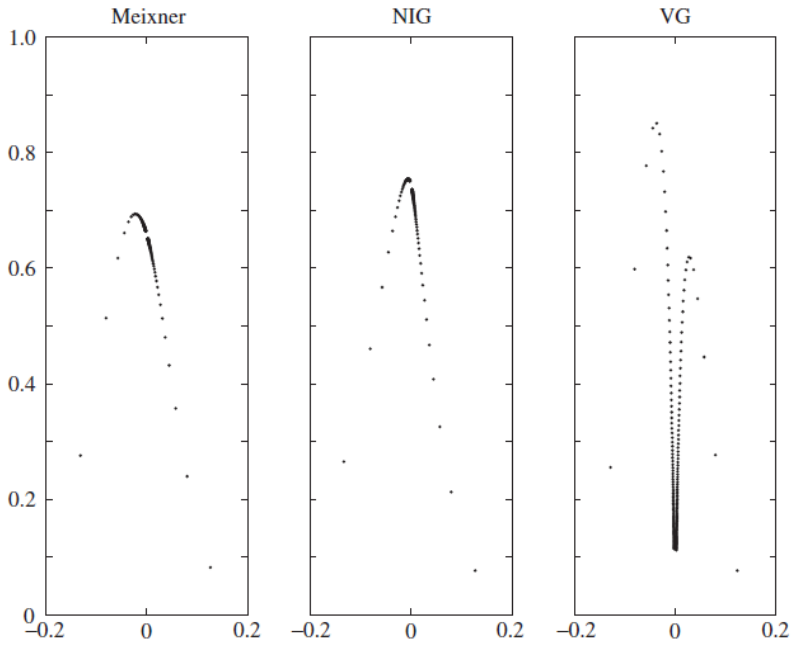

Figure 8.3 $a_{i-1} = -0.2/i$ and $a_{2k+2-i} = 0.2/i$, $1 \leqslant i \leqslant k+1$.

Simulation of a Gamma process

- To simulate a Gamma process: use a Gamma random number generator.
- When X is Gamma(a, b), then, for $c > 0$, X/c is Gamma(a, bc). $\Longrightarrow$ we only need a generator for Gamma$(a, 1)$ random numbers.
- Johnk's Gamma Generator (for $a \leq 1$):
  1. Generate two independent uniform random number u_1 and u_2.
- 2. Set $x = u_1^{1/a}$ and $y = u_2^{1/(1-a)}$
- 3. If $x + y \leq 1$ goto step 4, else goto step 1.
- 4. Generate an $Exp(1)$ random variable, i.e.$z = -log(u)$, where u is a uniform random number.
- 5. Return the number $zx/(x+y)$ as the Gamma$(a, 1)$ random number.
- There are other Gamma random generators (Berman generator and others).

Simulation of a Gamma process

- In order to simulate a sample path of a Gamma process $G = \{G_t, t \geq 0\}$, where G_t follows a Gamma(at, b) distribution, we simulate the value of G at time points $\{n\Delta t, n = 0, 1, ...\}$ as follows:
  (1) generate independent Gamma(at, b) random numbers $\{g_n, n \geq 1\}$.
  (2) Then:

$$G_0 = 0,$$
$$G_{n\Delta t} = G_{(n-1)\Delta t} + g_n, \quad n \geq 1.$$

Simulation of a Gamma process

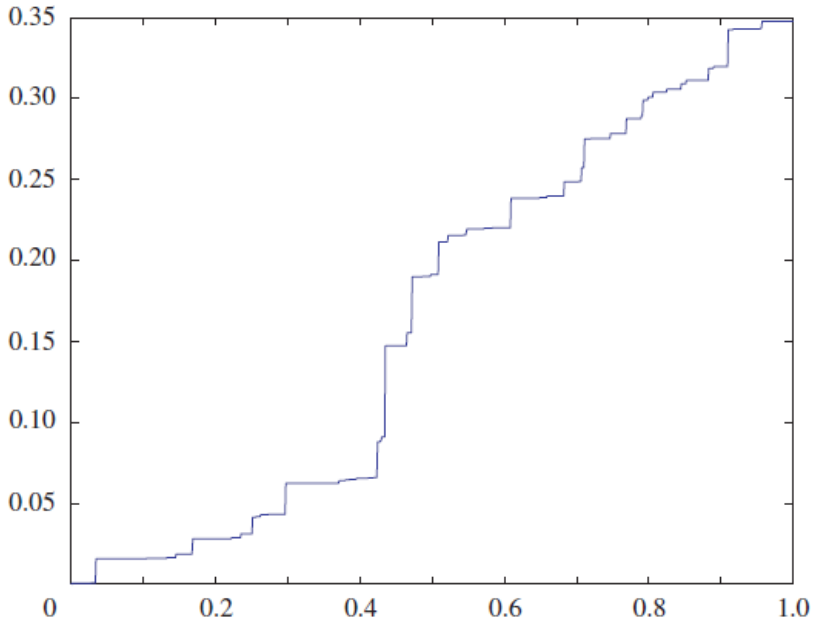

Figure 8.4 A sample path of a Gamma process.

Simulation of the Variance-Gamma (VG) process

- Simulation of VG with parameters $C, G, M > 0$ as the difference of 2 Gamma processes: $X_t^{(VG)} = G_t^{(1)} - G_t^{(2)}$,where $G^{(1)}$ has parameters $a = C$ and $b = M$, and $G^{(2)}$ has parameters $a = C$ and $b = G$.
- Simulation of a VG Process as a time-changed Brownian Motion: it is best explained if we use the parameters (σ, ν, ϕ) for the VG instead of C, G, M.

$$X_t^{(VG)} = \theta G_t + \sigma B_{G_t}.$$

A sample path of the VG process can be obtained by sampling a standard Brownian motion and a Gamma process.

Simulation of the Variance-Gamma (VG) process

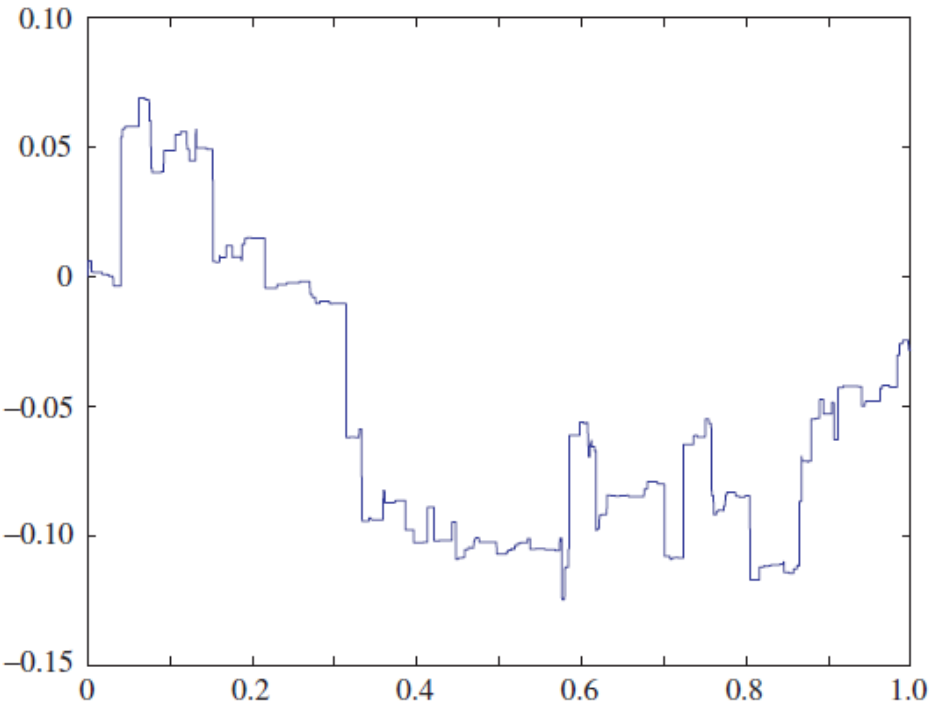

Figure 8.5 A sample path of a VG process.

Simulation of the Variance-Gamma (VG) process

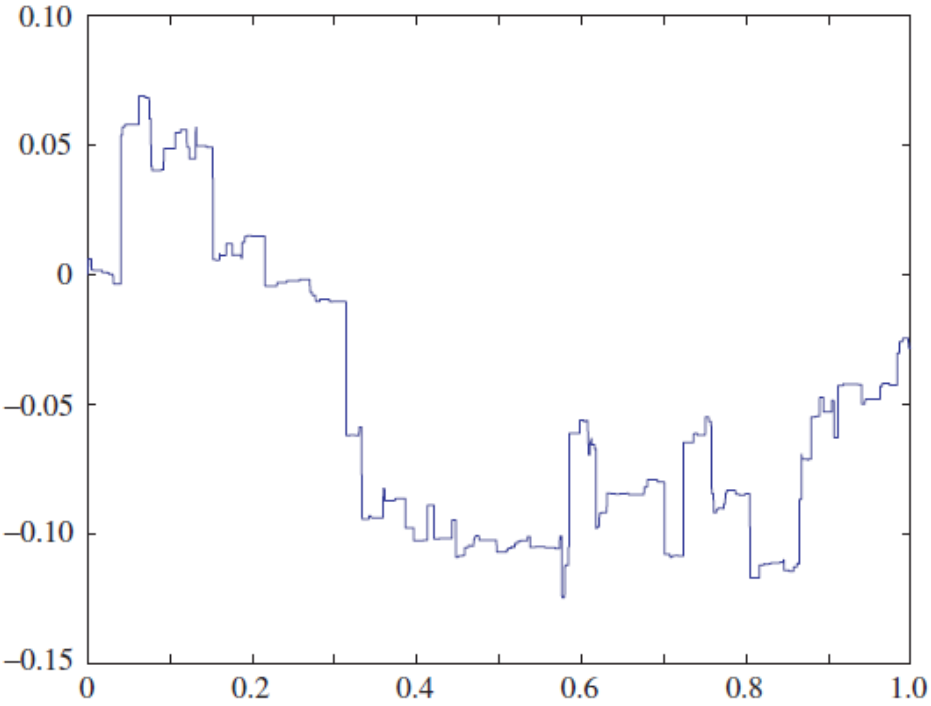

Figure 8.5 A sample path of a VG process.

Cont, R. and Tankov, P. (2003). Financial modelling with jump processes. Chapman and Hall/CRC Press.

Schoutens, W. (2002). Lévy Processes in Finance: Pricing Financial Derivatives. Wiley.