Master in Actuarial Science, Exam 24/01/2012. **2h30m**
Ratemaking and Experience Rating, 2nd year, 1st semester

1. Consider a motor insurance portfolio where drivers are of different types identified by the parameter θ, and labeled as 1, 2 and 3. Per year each risk in the portfolio can produce 0, 1, or 2 claims. Probabilities are shown in the table below. Suppose that a risk type is chosen at random, and two risks are chosen with replacement from that class. Suppose that a total of 2 claims was observed from those two risks drawn. Two more risks are then drawn with replacement from the same type, and it is of interest to predict the total on these next two. Let X be the total of claims from the two risks drawn and $f_X(.)$ its probability function.

Type	0	1	2
1	0.40	0.35	0.25
2	0.25	0.10	0.65
3	0.50	0.15	0.35

   (a) Determine the probability function $\pi(\theta)$ of the random variable Θ. $\pi(\theta_i) = 1/3, i = 1, 2, 3$.

   (b) Determine the conditional probability function $f_{X|\theta}(x|\theta = i)$ for the totals of claims for the two risks drawn when $i = 1, 2, 3$.

i	1	2	3
$\Pr(X = 0\|\theta_i)$	0.1600	0.0625	0.2500
$\Pr(X = 1\|\theta_i)$	0.2800	0.0500	0.1500
$\Pr(X = 2\|\theta_i)$	0.3225	0.3350	0.3725
$\Pr(X = 3\|\theta_i)$	0.1750	0.1300	0.1050
$\Pr(X = 4\|\theta_i)$	0.0625	0.4225	0.1225
Total	1.0	1.0	1.0

   (c) Calculate the probability of the total claims of the first two risks drawn equals 2:

$$\Pr(X_1 = 2) = \frac{1}{3}(0.3225 + 0.3350 + 0.3725) = 0.34333$$

   (d) Calculate the posterior distribution $\pi_{\Theta|X_1}(\theta|2)$.

i	$\Pr(X_1 = 2\|\theta_i)$	$\Pr(\Theta = \theta_i\|X_1 = 2) = \frac{\Pr(X_1=2\|\Theta=\theta_i)(1/3)}{0.34333}$
1	0.3225	0.313107
2	0.3350	0.325243
3	0.3725	0.361650

$0.313107 + 0.325243 + 0.361650 = 1.0$

   (e) Determine the conditional distribution $f_{X_2|X_1}(x_2|2)$ of the total X_2 of the next double draw given that $X_1 = 2$ was observed in the previous double draw.

x_2	$\Pr(X_2 = x_2\|X_1 = 2) = \sum \Pr(X_2 = x_2\|\Theta = \theta_i\|) \Pr(\Theta = \theta_i\|X_1 = 2)$
0	$0.16\,(0.313107) + 0.0625\,(0.325243) + 0.25\,(0.361650) = 0.160\,84$
1	$0.28\,(0.313107) + 0.05\,(0.325243) + 0.15\,(0.361650) = 0.158\,18$
2	$0.3225\,(0.313107) + 0.335\,(0.325243) + 0.3725\,(0.361650) = 0.344\,65$
3	$0.175\,(0.313107) + 0.13\,(0.325243) + 0.105\,(0.361650) = 0.135\,05$
4	$0.0625\,(0.313107) + 0.4225\,(0.325243) + 0.1225\,(0.361650) = 0.201\,29$

$1.0 = 0.160\,84 + 0.158\,18 + 0.344\,65 + 0.135\,05 + 0.201\,29$

   (f) Determine the Bayesian premium $E(X_2|X_1 = 2)$.

$$\mu(\theta_1) = 0\,(0.1600) + 1\,(0.2800) + 2\,(0.3225) + 3\,(0.1750) + 4\,(0.0625) = 1.7$$
$$\mu(\theta_2) = 0.0500 + 2\,(0.3350) + 3\,(0.1300) + 4\,(0.4225) = 2.8$$
$$\mu(\theta_3) = 0.1500 + 2\,(0.3725) + 3\,(0.1050) + 4\,(0.1225) = 1.7$$

$$E(X_2|X_1 = 2) = E(\mu(\Theta)\,|X_1 = 2) = 1.7\,(0.313107) + 2.8\,(0.325243) + 1.7\,(0.361650)$$
$$= 2.057\,767 \text{ , or}$$
$$E(X_2|X_1 = 2) = 0\,(0.160\,84) + 1\,(0.158\,18) + 2\,(0.344\,65) + 3\,(0.135\,05) + 4\,(0.201\,29)$$

(g) Compute the structural parameters $\mu = E(\mu(\Theta))$, $\upsilon = E(\upsilon(\Theta))$ and $a = V(\mu(\Theta))$.

$$\begin{aligned}
\mu &= \frac{1}{3}(1.7 + 2.8 + 1.7) = 2.0666\,7 \\
& \quad \upsilon(\theta_1) = 1.255 \quad \upsilon(\theta_2) = 1.480 \quad \upsilon(\theta_3) = 1.655 \\
\upsilon &= \frac{1}{3}(1.255 + 1.48 + 1.655) = 1.463\,3 \\
a &= \frac{1}{3}\left(1.7^2 + 2.8^2 + 1.7^2\right) - 2.0666\,7^2 = 4.54 - 2.066\,67^2 = 0.268\,889
\end{aligned}$$

(h) Compute Bühlmann's credibility premium.

$$\begin{aligned}
k &= 1.463\,3/0.268\,889 = 5.442 \\
z &= \frac{1}{1 + 1.463\,3/0.268\,889} = 0.155\,228 \\
P_c &= 0.155\,228\,(2) + (1 - 0.155\,228)\,2.0666\,7 = 2.056\,3
\end{aligned}$$

(i) On what condition(s) can we talk on *exact credibility model*? Comment appropriately.
If Bayesian premium equals Bühlmann's credibility premium, i.e., if Bayesian premium is linear as Bhulmann's is the best linear estimator under the same criteria.

2. Retrieve Problem 1. The insurer uses a *bonus* system based on claim frequency with three classes (Class 1, 2, and 3) in increasing order of riskiness. Transition rules are the following: A policy with one year without claims goes to the previous class at the beginning of the following year, unless it is already in Class 1 where it remains. If a policy reports a claim goes to Class 3, unless it is already there where stays.

Let λ be the probability of a policy get a claim (one or more) taken at random from the portfolio. Class 2 is the entry class and *premia* vector (in €) is given by $\mathbf{b} = (150, 225, 300)$.

(a) Consider a policy with risk parameter θ.

i. Set the transition rules matrix and determine the 1-step transition probabilities matrix (make calculations as function of θ);

$$T = \begin{bmatrix} \{0\} & & \{1,2,...\} \\ \{0\} & & \{1,2,...\} \\ & \{0\} & \{1,2,...\} \end{bmatrix}; \; P = \begin{bmatrix} 1-\lambda(\theta) & & \lambda(\theta) \\ 1-\lambda(\theta) & & \lambda(\theta) \\ & 1-\lambda(\theta) & \lambda(\theta) \end{bmatrix}$$

ii. Determine the limit distribution;
Note that at step-2 we already get the limit distribution:

$$P^2 = \begin{bmatrix} [1-\lambda(\theta)]^2 & \lambda(\theta)(1-\lambda(\theta)) & \lambda(\theta) \\ [1-\lambda(\theta)]^2 & \lambda(\theta)(1-\lambda(\theta)) & \lambda(\theta) \\ [1-\lambda(\theta)]^2 & \lambda(\theta)(1-\lambda(\theta)) & \lambda(\theta) \end{bmatrix}$$

We could get it solving for (x, y, z):

$$\begin{cases} (1-\lambda(\theta))\,x + (1-\lambda(\theta))\,y = x \\ (1-\lambda(\theta))\,z = y \\ x + y + z = 1 \end{cases}$$

iii. For a policy entering in the system, what is the probability of belonging to Class 2 after two years;
Let Y_n be the state of the process in year n

$$\begin{aligned}
P[Y_2 = 2] &= \sum_{1,2,3} P[Y_2 = 2 | Y_0 = i]\, P[Y_0 = i] \\
&= P[Y_2 = 2 | Y_0 = 2] = \lambda(\theta)(1-\lambda(\theta))
\end{aligned}$$

iv. A certain policyholder who considers himself to be a Type 1 driver is placed is *Bonus* Class 1, in a given year. In that year he had an accident with a correspondent claim amount x. He can report the claim to the insurer, however that would imply a higher premium for the following years. Discuss the situations by making appropriate, but simple, calculations.
Possible calculations:

- If he is in Class 1 he payed in the year 150
- He does not report, so pays a repair amount of x and then premium 150: $x + 150$;
- If he reports he pays premium next year of 300; in the best scenario, he pays following year 225; he can return to Class 1 after 2 two years (at least). For instance, In the best scenario we can equate

$$x + 2 \times 150 = 300 + 225 \Leftrightarrow x = 225.$$

  He could take a repair of 225.
- He could equate based on the average premium for instance, like (N is number of claims per year)

$$\begin{aligned} x + 150 &= 300 + 225 \Pr[N = 0|T_1] + 300 \Pr[N \geq 1|T_1] \\ x &= 300 + 225\,(0.4) + 300\,(0.6) - 150 = 420 \end{aligned}$$

v. For a driver with risk parameter $\theta = 2$ calculate the probability function of the premium payment one year after having entered the system. Compute also the average premium.

$$\begin{aligned} P[Y_1 = 1|\theta = 2] &= P[Y_1 = 1|Y_0 = 2, \theta = 2] = 0.25 \\ P[Y_1 = 2|\theta = 2] &= P[Y_1 = 2|Y_0 = 2, \theta = 2] = 0 \\ P[Y_1 = 3|\theta = 2] &= P[Y_1 = 3|Y_0 = 2, \theta = 2] = 0.75 \\ \text{Average Premium} &: 150(0.25) + 300(0.75) = 262.5 \end{aligned}$$

(b) Suppose that the stationary distribution for a given θ is given by vector $\left([1-\lambda(\theta)]^2; [1-\lambda(\theta)]\lambda(\theta); \lambda(\theta)\right)$. Calculate the distribution for a risk taken at random from the portfolio. Compute also the average premium.

$$\begin{aligned} E[\lambda(\theta)] &= \frac{1}{3}(0.6 + 0.75 + 0.5) = 0.616\,67 \\ E\left[\lambda(\theta)^2\right] &= \frac{1}{3}\left(0.6^2 + 0.75^2 + 0.5^2\right) = 0.390\,83 \\ E\left([1-\lambda(\theta)]^2\right) &= \frac{1}{3}\left((1-0.6)^2 + (1-0.75)^2 + (1-0.5)^2\right) = 0.157\,5 \\ \Pr[P = 150] &= 0.157\,5 \\ \Pr[P = 225] &= 0.616\,67 - 0.390\,83 = 0.225\,84 \\ \Pr[P = 300] &= 0.616\,67 \\ 1.0 &= 0.157\,5 + 0.225\,84 + 0.616\,67 \\ \text{Premium Mean} &= 0.157\,5(150) + 0.225\,84(225) + 0.616\,67(300) = 259.44 \end{aligned}$$

3. Suppose you work as an actuary for an insurance company and you need to define a new tariff for the third party liability in motor insurance to put in place in the beginning of 2012.

(a) Consider the *pros* and *cons* of using the available data of 2011 as an alternative to those of 2010.
Those of 2010 may not yet be stabilized yet, those of 2011 may be *up to date*, however may not be complete yet, fully reported....

(b) As usual, we model separately the claim frequency and the expected cost per unit. For the expected cost we decided to consider 5 rating factors:

- Driver's age (1=until 20 years of age, 2= from 21-25, 3 = from 26-35, 4= from 36-50 and 5=more than 50)
- Experience (1=2 or less license years , 2=more than 2)
- Region (1=Lisbon urban area, 2=Oporto urban area, 3=South, 4=Centre and 5=North)
- Capital insured (1 = minimum legal, 2 = medium, 3 = Insurer's maximum).
- Power of vehicle (1=till 75 HP, 2 = 75-120 HP and 3= more than 120HP)

Consider the model shown in the Annex.

i. Argue the use of the model and refer possible changes to be made. Use existing information when available.
At least, 'CILevel' is not significant because of a high p-level. Also, Region3 is questionable. We should do the join null test for 'CILevel', not the separate null tests (data offer only those)...

ii. What is the size of the portfolio on which the model is based?
16353+1.

iii. What is the ratio between the larger and the smaller expected claim cost?
Max: $\exp\{7.287927 + 0.087757 + 0.128912 + 0.006322 + 0.148721\} \simeq 2120.9768$
Min: $\exp\{7.287927 - 0.254591\} \simeq 1133.7998$. Ratio: $2120.9768/1133.7998 = 1.8707$

iv. Suppose now a policy from the Centre Region, vehicle with 90HP, Driver with 53 years of age and 30 years of Experience, CI minimum, and expected frequency of 0.1. What will be the corresponding pure premium?

$$0.1 \exp\{7.287927 - 0.254591 + 0.087757 + 0.128912 + 0.050938\}$$