Ratemaking and Experience Rating

Master on Actuarial Science

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5 Ratemaking and GLM

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Introduction and Concepts

Ratemaking:

- "Pricing" insurance, calculation of Insurance *Premia*
- Building a **tariff** for a portfolio, or portfolios somehow connected

Experience rating: adjust future premiums based on past experience

Insurance **Premium**: Price for buying insurance (for a period). 2 components:

- Economic criteria: *market price*, admin costs
- Actuarial criteria:
  - based on technical aspects of the risk
  - Meant to cover future claims
  - We only consider this here

- **Tariff**:
    - System of premiums for the risks of a portfolio (homogeneous)
    - Sets a base premium (homogeneous)
    - plus a set of bonus/malus (heterogeneous)
- **Exposure**: Risk volume, in risk units, no.
- **Risk unit**: policy
- **Claim**: an accident generates a claim
- **Claim frequency**: number of claims, distribution
- **Severity**: amount of the claim
- Loss reserving
- **Pure premium**: Risk mean, loss mean
- **Loss ratio**: paid claims/premiums

Credibility formula

Let X be a given risk in a portfolio, with Pure Premium $E(X)$, unknown:

- If the risk is has been sufficiently observed

$$E(X) \simeq \overline{X} \quad (\textit{Full Credibility})$$

- If not, use *Partial Credibility*, **Credibility Formula**:

$$\begin{aligned} E(X) &\simeq z\overline{X} + (1-z)M \\ z &= \frac{n}{n+k} \end{aligned}$$

- Credibility factor: z, $0 \leq z < 1$
- n: No. observations; k: some positive constant
- M: Externally obtained mean (*Manual rate*).

Example

For a given risk $X|\theta \frown Bin(1;\theta)$, obs'd 10 yrs, 20 risks. $\bar{X} = 1.45$.

Ano i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1			1								1						1	1		
2							1		1		1						1			
3			1				1		1	1										
4																	1			
5									1		1									
6						1			1			1								
7									1					1			1			
8												1								
9						1					1		1							
10									1			1					1		1	
$\hat{\theta}_j$	0,0	0,0	0,2	0,0	0,0	0,2	0,2	0,0	0,6	0,1	0,4	0,3	0,1	0,1	0,0	0,0	0,5	0,1	0,1	0,0

"Limited Fluctuation" and "Greatest Accuracy" theories

1. *Limited Fluctuation*:
   1. From some computed $n : n_0$ use *Full* credibility;
   2. Otherwise: Use *Partial* credibility. But what M, k?
2. *Greatest Accuracy*: Bayesian approach.

Example (Ex. 20.9)

Two types of drivers: Good and Bad. Good are 75% of the population and in one year have have 0 claims w.p. 0.7, 1 w.p. 0.2 and 2 w.p. 0.1. Bad drivers, respectively, 25%, 0.5, 0.3, 0.2. when a driver buys insurance insurer does not know it's category. We assign an unknown risk parameter, θ.

Joint and conditional distribution and expectation

Example (Ex. 20.9 cont.)

x	$P(X=x\|\theta=G)$	$P(X=x\|\theta=B)$	θ	$P(\Theta=\theta)=\pi(\theta)$
0	0.7	0.5	G	0.75
1	0.2	0.3	B	0.25
2	0.1	0.2		

Bivariate random variable: (X,Y). D.f. $F_{X,Y}$, pdf or pf $f_{X,Y}$

- $f_{X,Y}(x,y)$, marginals f_X, f_y. If independent: $f_{X,Y}=f_X f_Y$.
- Conditional (Conditional ind.: $f_{X,Y|Z}=f_{X|Z}f_{Y|Z}$):

$$f_{X|Y}(x)=\frac{f_{X,Y}(x,y)}{f_Y(y)} \qquad f_{Y|X}(y)=\frac{f_{X,Y}(x,y)}{f_X(x)}$$

$$f_{X,Y}(x,y)=f_{X|Y}(x)f_Y(y) \qquad f_{X,Y}(x,y)=f_{Y|X}(y)f_X(x)$$

- Marginals

$$f_X(x) = \int f_{X,Y}(x,y)dy \qquad f_Y(y) = \int f_{X,Y}(x,y)dx$$
$$f_X(x) = f_{X|Y}(x)f_Y(y)dy \qquad f_Y(x) = f_{Y|x}(x)f_X(y)dx$$

- Expectations, Iterated expectation

$$E[E(X|Y)] = E[X];\ E[E(Y|X)] = E[Y]$$
$$V[X] = E[V(X|Y)] + V[E(X|Y)]$$
$$Cov[X,Y] = E[Cov(X,Y|Z)] + Cov[E(X|Z)E(Y|Z)]$$

Example (Ex. 20.9 cont'd)

Suppose we observed for a particular risk: $\mathbf{X} = (X_1, X_2) = (0;1)$. Given θ obs are independent.

$$
\begin{aligned}
f_{\mathbf{X}}(0,1) &= \sum_{\theta} f_{\mathbf{X}|\boldsymbol{\theta}}(0,1|\theta)\pi(\theta) = \sum_{\theta} f_{X_1|\theta}(0|\theta) f_{X_2|\theta}(1|\theta)\pi(\theta) \\
&= 0.7(0.2)(0.75) + 0.5(0.3)(0.25) = 0.1425 \\
f_{\mathbf{X}}(0,1,x_3) &= \sum_{\theta} f_{\mathbf{X},\mathbf{X_3}|\boldsymbol{\theta}}(0,1,x_3|\theta)\pi(\theta) \\
&= \sum_{\theta} f_{X_1|\theta}(0|\theta) f_{X_2|\theta}(1|\theta) f_{X_3|\theta}(x_3|\theta)\pi(\theta) \\
f(0,1,0) &= 0.09995;\ f(0,1,1) = 0.003225;\ f(0,1,2) = 0.01800 \\
f(0|0,1) &= 0.647368;\ f(1|0,1) = 0.226316;\ f(2|0,1) = 0.126316 \\
\pi(G|0,1) &= 0.736842;\ \pi(B|0,1) = 0.263158
\end{aligned}
$$

Joint and conditional distribution and expectation

Example (Ex. 20.11)

Let $X|\theta \frown Poisson(\theta)$ and
$\Theta \frown Gamma(\alpha, \beta) \Rightarrow X \frown NBinomial(\alpha, \beta)$

$$E(X|\theta) = \theta \Rightarrow E(X) = E(E(X|\Theta)) = E(\Theta) = \alpha\beta$$
$$V(X|\theta) = \theta \Rightarrow V(X) = V(E(X|\Theta)) + E(V(X|\Theta)) = \alpha\beta\,(1+\beta)$$

Example (Ex. 20.10)

Let $X|\theta \frown \exp(1/\theta)$, mean $1/\theta$, and $\Theta \frown Gamma(4, 0.001)$.

$$f(x|\theta) = \theta e^{-\theta x},\ x,\theta > 0$$
$$\pi(\theta) = \theta^3 e^{-1000\theta} 1000^4/6,\ \theta > 0$$

Example (Ex. 20.10)

Suppose a risk had 3 claims of 100, 950, 450.

$$\begin{aligned} f(100,950,450) &= \int_0^\infty f(100,950,450|\theta)d\pi(\theta)d\theta \\ &= \int_0^\infty f(100|\theta)f(950|\theta)f(450|\theta)d\pi(\theta)d\theta \\ &= \frac{1,000^4}{6}\frac{720}{2,500^7} \end{aligned}$$

Similarly

$$f(100,950,450,x_4) = \frac{1,000^4}{6}\frac{5040}{(2,500+x_4)^8}$$

Example (Ex. 20.10)

Preditive density, posterior

$$f(x_4|100,950,450) = \frac{7\,(2500)^7}{(2,500+x_4)^8} \rightarrow Pareto(7;2500)$$

$$\pi(\theta|100,950,450) = \theta^6 e^{-2500\theta} 2500^7/\Gamma(7) \rightarrow Gamma(7;1/2500)$$

$$\begin{aligned}
\mu_4(\theta) &= E(X_4|\theta) =? \\
E(X_4|100,950,450) &= 416,67 \\
\mu &= E(X_4) = E(1/\Theta) = 1000/3 = 333.3(3) \\
\bar{X} &= 500 \\
\mu &< E(X_4|100,950,450) < \bar{X}
\end{aligned}$$

Bayesian approach

Let a portfolio of risks, homogeneous, but "different":

- Homogeneous: risks follow the same distribution family
- Heterogeneous: distribution parameter is different.

A **given risk** comes attached with a paramenter θ:

- Fixed, but unknown, not observable;
- Only claims are observed: $(X_1, X_2, ..., X_n) = \mathbf{X}$;
- θ is the hidden aspects of the risk, which differs from others;
- Like classical statistics: Use past data $\mathbf{X}$ to predict X_{n+1}
- **Risk (pure) Premium**: $E(X_{n+1}|\theta) = \mu_{n+1}(\theta)$.
- Opposed to **Collective (pure) Premium**: $E(X_{n+1}) = \mu_{n+1}$.

Hypothesis

1. Given θ, $X_1|\theta, X_2|\theta, ..., X_n|\theta, X_{n+1}|\theta$ are (conditionally) independent.
   θ is realization of a random variable: $\Theta \frown \pi(\theta)$
2. The different risks in the portfolio are independent.

Premium for the next year:

- Risk Premium: $E(X_{n+1}|\theta) = \mu_{n+1}(\theta)$. Unknown.
- Collective Premium: $E(E(X_{n+1}|\theta)) = \mu_{n+1}$. In general $\mu_{n+1}(\theta) \neq \mu_{n+1}$
- Bayesian premium (mean of the preditive dist.):

$$\begin{aligned} E(X_{n+1}|\mathbf{X}) &= \int x f_{X_{n+1}|\mathbf{X}}(x|\mathbf{x})dx \\ &= \int \mu_{n+1}(\theta)\pi_{\Theta|\mathbf{X}}(\theta|\mathbf{x})d\theta \end{aligned}$$

The Credibility Premium

Let the estimator $\widetilde{\mu_{n+1}}(\theta)$ be of linear form: $\alpha_0 + \sum_{j=1}^{n} \alpha_j X_j$:

$$\min Q = E\left\{ \left[\mu_{n+1}(\theta) - \left(\alpha_0 + \sum_{j=1}^{n} \alpha_j X_j \right) \right]^2 \right\}$$

Solution: Find $\alpha_0, \alpha_1, ..., \alpha_n$:

$$\frac{\partial}{\partial \alpha_0} Q = -2E\left\{ \mu_{n+1}(\theta) - \left(\alpha_0 + \sum_{j=1}^{n} \alpha_j X_j \right) \right\} = 0$$

$$\frac{\partial}{\partial \alpha_i} Q = -2E\left\{ \left[\mu_{n+1}(\theta) - \left(\alpha_0 + \sum_{j=1}^{n} \alpha_j X_j \right) \right] X_i \right\} = 0,\ i = 1, ..., n$$

θ, $X_1, X_2, ..., X_n, X_{n+1}$ are all random variables.

The Credibility Premium. Normal equations

$$E(X_{n+1}) = \widetilde{\alpha}_0 + \sum_{j=1}^{n} \widetilde{\alpha}_j E\left[X_j\right] = E\left(\widetilde{\mu_{n+1}}(\theta)\right), \text{ unbiasedness eq.}$$

$$Cov(X_i, X_{n+1}) = \sum_{j=1}^{n} \widetilde{\alpha}_j Cov\left[X_i, X_j\right], \; i = 1, ..., n.$$

$$
\begin{aligned}
\min Q &= \min E\left\{\left[\mu_{n+1}(\theta) - \left(\alpha_0 + \sum_{j=1}^{n} \alpha_j X_j\right)\right]^2\right\} \\
&= \min E\left\{\left[E\left[X_{n+1}|\mathbf{X}\right] - \left(\alpha_0 + \sum_{j=1}^{n} \alpha_j X_j\right)\right]^2\right\} \\
&= \min E\left\{\left[X_{n+1} - \left(\alpha_0 + \sum_{j=1}^{n} \alpha_j X_j\right)\right]^2\right\}
\end{aligned}
$$

Obs: $E\left[X_{n+1}\right] = E\left[X_{n+1}|\mathbf{X}\right] = E\left[E\left[X_{n+1}|\Theta\right]\right] = E\left[\mu_{n+1}(\Theta)\right]$; $\mu_{n+1}(\theta) = E\left[X_{n+1}|\boldsymbol{\theta}\right]$.

Addition to **H**ypothesis **1**

1. Given θ, $X_1|\theta, X_2|\theta, ..., X_n|\theta, X_{n+1}|\theta$ have the same mean and variance:

$$\begin{aligned} \mu(\theta) &= E\left(X_j|\theta\right) \\ v(\theta) &= Var\left(X_j|\theta\right). \end{aligned}$$

Let

$$\mu = E\left[\mu(\theta)\right], \; v = E\left[v(\theta)\right], \; a = Var\left[\mu(\theta)\right]$$

Solution:

$$\begin{aligned} \widetilde{\alpha}_0 + \sum_{j=1}^{n} \widetilde{\alpha}_j X_j &= z\overline{X} + (1-z)\mu \\ z &= \frac{n}{n+k} \\ k &= v/a \end{aligned}$$

1. z: called Bühlmann's credibility factor
2. Credibility premium is a weighted average from $\overline{X}$ and μ.
3. $z \longrightarrow 1$ when $n \longrightarrow \infty$, more credit to sample mean
4. If portfolio is fairly homogeneous w.r.t. Θ, then $\mu(\Theta)$ does not vary much, hence small variability.
   Thus a is small relative to $v \longrightarrow k$ is large, z is closer to 0
5. Conversely, if the portfolio is heterogeneous, z is closer to 1
6. Bühlmann's model is the simplest credibility model, no change over time

Proof: Estimator proposed: $\hat{m}_j = \alpha + \beta \overline{X}_{.j}$, so that

$$\min R = \min \mathrm{E}\left[\left(\mu(\theta_j) - \hat{m}_j\right)^2\right] = \min \mathrm{E}\left[\left(\mu(\theta_j) - \alpha - \beta \overline{X}_{.j}\right)^2\right].$$

Set

$$\mathrm{E}\left[\left(\left(\mu(\theta_j) - \beta \overline{X}_j]\right) - \alpha\right)^2\right] = \mathrm{V}[\mu(\theta_j) - \beta \overline{X}_j] + \left(\mathrm{E}\left[\mu(\theta_j) - \beta \overline{X}_{.j}\right] - \alpha\right)^2$$

Minimizing a^*:

$$\begin{aligned} \alpha^* &= \mathrm{E}[\mu(\theta_j) - b^* \overline{X}_{.j}] = \mathrm{E}[\mu(\theta_j)] - b\,\mathrm{E}[\overline{X}_{.j}]. \\ \alpha^* &= (1 - \beta^*)\,\mathrm{E}[\mu(\theta_j)], \text{ since} \\ \mathrm{E}[\overline{X}_{.j}] &= \mathrm{E}[\mathrm{E}[\overline{X}_{.j} | \theta_j]] = \mathrm{E}[\mu(\theta_j)] \end{aligned}$$

The other part

$$
\begin{aligned}
\mathrm{V}[\mu(\theta_j) - \beta \overline{X}_{.j}] &= \mathrm{E}[\mathrm{V}[\mu(\theta_j) - \beta \overline{X}_{.j}|\theta_j]] + \mathrm{V}[\mathrm{E}[\mu(\theta_j) - \beta \overline{X}_{.j}|\theta_j]] \\
&= \beta^2 \mathrm{E}[v(\theta)] + (1-\beta)^2 \mathrm{V}[\mu(\theta_j)]. \\
&= \frac{\beta^2}{n} v + (1-\beta)^2 a. \\
\mathrm{V}[\overline{X}_{.j}|\theta_j] &= \frac{1}{n} \mathrm{V}[X_{ij}|\theta_j]
\end{aligned}
$$

Differentiating w.r.t. β and equating,

$$
\frac{2\beta}{n} v - 2(1-\beta)a = 0 ,
$$

$$
\beta^* = \frac{a}{a + \frac{1}{n} v}
$$

Example (Ex.20.9 cont'd)

$$\mu_3(G) = 0.4 \qquad \mu_3(B) = 0.7$$
$$E[X_3|0,1] = 0.478948 \qquad \mu_3 = 0.475 \quad \bar{X} = 0.5$$
$$a = V[\mu(\theta)] = 0.016875 \qquad v = E[v(\theta)] = 0.4825$$
$$k = v/a = 28.5926 \qquad z = 2(2+k)^{-1} = 0.0654$$
$$z\overline{X} + (1-z)\mu = 0.0654(0.5) + 0.9346(0.475) = 0.4766$$

Example (Ex. 20.10)

Exact credibility example.

$$\begin{aligned} E(X_4|100,950,450) &= 416,67; \quad \bar{X} = 500 \\ \mu = E(X_4) = E(1/\Theta) &= 1000/3 = 333.3(3) \\ z\overline{X} + (1-z)\mu &= E(X_4|100,950,450). \end{aligned}$$

Changes to **H**ypothesis **1** in Bühlmann's model:

1. Given θ, $X_1|\theta, X_2|\theta, ..., X_n|\theta, X_{n+1}|\theta$ have the same mean, variance:

$$E(X_j|\theta) = \mu(\theta) \ (same)$$
$$Var(X_j|\theta) = \frac{\upsilon(\theta)}{m_j}.$$

- m_j is some known constant measuring exposure
- Ex: group insurance where its size changes
- Initially, the model was first presented for reinsurance.
- $Var(X_j) = E[Var(X_j|\theta)] + Var[E(X_j|\theta)] = \frac{\upsilon}{m_j} + a$

Solution:

$$P_c = \widetilde{\alpha}_0 + \sum_{j=1}^{n} \widetilde{\alpha}_j X_j = z\overline{X} + (1-z)\mu$$

$$z = \frac{m}{m+k} \qquad k = v/a$$

$$\overline{X} = \sum_{j=1}^{n} \frac{m_j}{m} X_j \qquad m = \sum_{j=1}^{n} m_j \text{ (total exposure)}$$

- Factor z depends on m (total exposure)
- $\overline{X}$ is a weighted average, m_j/m is the weight
- $m_j X_j$ is the total loss of the group in year j
- (Total) Credibility premium for the group, next year : $m_{n+1}\left[z\overline{X} + (1-z)\mu\right]$

Example (Ex.20.19)

N_j: No. of claims in year j for a group policy holder with risk parameter and m_j individuals. $N_j \frown Poisson(m_j\theta)$. Let $X_j = N_j/m_j$. $\Theta \frown Gamma(\alpha, \beta)$.

$$
\begin{aligned}
E(X_j|\theta) &= \mu(\theta) = \theta;\ V(X_j|\theta) = V(N_j/m_j|\theta) = \frac{v(\theta)}{m_j} = \frac{\theta}{m_j} \\
\mu &= E(\Theta) = \alpha\beta;\ a = V(\Theta) = \alpha\beta^2;\ v = E(\Theta) = \alpha\beta. \\
k &= v/a = 1/\beta;\ z = \frac{m\beta}{m\beta+1} \\
P_c &= \frac{m\beta}{m\beta+1}\overline{X} + \frac{1}{m\beta+1}\alpha\beta
\end{aligned}
$$

Example (Ex.20.19)

N_j: No. of claims in year j for a group policy holder with risk parameter θ and m_j individuals, $j = 1, ..., n$. $N_j \frown Poisson(m_j\theta)$. Let $X_j = N_j / m_j$. $\Theta \frown Gamma(\alpha, \beta)$. Bayesian premium (mean of the preditive dist.):

$$\begin{aligned} E(X_{n+1}|\mathbf{X}) &= \int x f_{X_{n+1}|\mathbf{X}}(x|\mathbf{x})dx = \int \mu_{n+1}(\theta)\pi_{\Theta|\mathbf{X}}(\theta|\mathbf{x})d\theta \\ &= E(E(X_{n+1}(\theta)|\theta, \mathbf{X})) = E((\mu_{n+1}(\theta)|\mathbf{X})) \\ &= E(\theta|\mathbf{X}) \end{aligned}$$

$$\begin{aligned} \Pr[N_j = x|\theta] &= (m_j\theta)^x e^{-m_j\theta}/x!;\ \pi(\theta) = \frac{\theta^{\alpha-1}e^{-\theta/\beta}}{\Gamma(\alpha)\beta^\alpha} \\ \pi_{\Theta|\mathbf{X}}(\theta|\mathbf{x}) &\propto \left[\prod_{i=1}^n f_{X_j|\mathbf{X}}(x_j|\mathbf{x})\right]\pi(\theta) \end{aligned}$$

Example (Ex.20.19)

N_j: No. of claims in year j for a group policy holder with risk parameter and m_j individuals, $j = 1, ..., n$. $N_j \frown Poisson(m_j\theta)$. Let $X_j = N_j/m_j$. $\Theta \frown Gamma(\alpha, \beta)$.

$$\Theta|\mathbf{x} \quad \frown \quad Gamma\left(\alpha_* = \alpha + \sum_{j=1}^{n} m_j x_j; \beta_* = (1/\beta + m)^{-1}\right)$$

$$E(X_{n+1}|\mathbf{X} = \mathbf{x}) = \alpha_*\beta_* = \frac{\alpha + \sum_{j=1}^{n} m_j x_j}{(1/\beta + m)}$$

$$= \frac{m\beta}{m\beta + 1}\overline{X} + \frac{1}{m\beta + 1}\alpha\beta = P_c$$

Credibility Premium, here consider $\mu_{n+1}(\theta) = \mu(\theta)$:

$$\widetilde{\mu_{n+1}}(\theta)\colon \min\left\{ Q = E\left\{ \left[\mu_{n+1}(\theta) - \left(\alpha_0 + \sum_{j=1}^{n} \alpha_j X_j \right) \right]^2 \right\} \right\}$$

If we don't impose a linear estimator, only some fucntion of $\mathbf{X}$, $m(\mathbf{X})$:

$$\overset{*}{m}(\mathbf{X})\colon \min\left(E\left\{ \left[\mu(\theta) - m(\mathbf{X}) \right]^2 \right\} = E\left[E\left\{ \left[\mu(\theta) - m(\mathbf{X}) \right]^2 | \mathbf{X} \right\} \right] \right)$$

or minimize

$$E\left\{ \left[\mu(\theta) - m(\mathbf{X}) \right]^2 | \mathbf{X} \right\} = V\left[\mu(\theta) | \mathbf{X} \right] + \left(E\left[\mu(\theta) | \mathbf{X} \right] - m(\mathbf{X}) \right)^2$$

$$\overset{*}{m}(\mathbf{X}) = E\left[\mu(\theta) | \mathbf{X} \right]$$

Bayes estimator, relative to the square loss function and prior $\pi(\theta)$

Exact Credibility: When $\widetilde{\mu_{n+1}}(\theta) = \overset{*}{m}(\mathbf{X}) = E[\mu(\theta)|\mathbf{X}]$, credibility Premium=Bayesian Prremium

Hipothesis: Changes to **H1** of Bühlmann's (stronger): $f_{X_j}(.|\theta) = f_X(.|\theta) \forall j$.

$$\begin{aligned}
\mathrm{E}[\mu(\theta)|\mathbf{X}] &= \int \mu(\theta)\pi(\theta|\mathbf{x})d\theta = \int \mu(\theta)\frac{f(\theta,\mathbf{x})}{f(\mathbf{x})}d\theta \\
&= \int \mu(\theta)\frac{f(\mathbf{x}|\theta)\pi(\theta)}{\int f(\mathbf{x}|\theta)\pi(\theta)}d\theta = \frac{\int \mu(\theta)\prod_{j=1}^{n} f(x_j|\theta)\pi(\theta)d\theta}{\int_\Theta \prod_{j=1}^{n} f(x_j|\theta)\pi(\theta)d\theta} \\
&= \frac{\int \mu(\theta)L(\theta)\pi(\theta)d\theta}{\int_\Theta L(\theta)\pi(\theta)d\theta};
\end{aligned}$$

$$\pi(\theta|\mathbf{x}) = \frac{L(\theta)\pi(\theta)}{\int_\Theta L(\theta)\pi(\theta)d\theta}$$

Example

For a given risk $X|\theta \frown Bin(1;\theta)$, $\Theta \frown U(\alpha,\beta)$, obs'd 10 yrs, 20 risks. $\bar{X} = 1.45$, $\mu_{n+1}(\theta) = \mu(\theta) = \theta$.

$$f(x|\theta) = \theta^x (1-\theta)^{1-x}, \quad x = 0,\, 1; \quad 0 < \theta < 1.$$

$$\pi(\theta) = \frac{1}{\beta-\alpha}, \quad 0 < \alpha < \theta < \beta < 1 \quad (\beta > \alpha)$$

$$\overset{*}{m}(\mathbf{x}) = \mathrm{E}[\theta|\mathbf{x}] = \frac{\sum_{k=1}^{n-n\bar{x}} (-1)^k \frac{\beta^{n\bar{x}+k+2} - \alpha^{n\bar{x}+k+2}}{(n-n\bar{x}-k)!k!(n\bar{x}+k+2)}}{\sum_{k=1}^{n-n\bar{x}} (-1)^k \frac{\beta^{n\bar{x}+k+1} - \alpha^{n\bar{x}+k+1}}{(n-n\bar{x}-k)!k!(n\bar{x}+k+1)}},$$

Example (*Beta-Binomial* model)

For a given risk $X|\theta \frown Bin(1;\theta)$, $\Theta \frown Beta(\alpha,\beta)$, α, $\beta > 0$, $\bar{X} = 1.45$

$$\pi(\theta) = \frac{\theta^{\alpha-1}(1-\theta)^{\beta-1}}{B(\alpha,\beta)};\ \theta\epsilon(0;1),\ B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$$

$$L(\theta) = \prod_{j=1}^{n} f(x_j|\theta) = \theta^{\sum_{j=1}^n x_j}(1-\theta)^{n-\sum_{j=1}^n x_j};$$

$$\pi(\theta|\mathbf{x}) = \frac{L(\theta)\pi(\theta)}{\int_0^1 L(\theta)\pi(\theta)d\theta} = \frac{\theta^{\sum_j x_j+\alpha-1}(1-\theta)^{n+\beta-\sum_j x_j-1}}{B(\sum_j x_j+\alpha;\ n+\alpha-\sum_j x_j)},$$

$$\pi(\theta|\mathbf{x}) \equiv Beta(\sum_j x_j+\alpha;\ n+\beta-\sum_j x_j)$$

$$\mathrm{E}[\theta|\mathbf{x}] = \frac{\sum_j x_j+\alpha}{\alpha+\beta+n} = \frac{n}{\alpha+\beta+n}\bar{x} + \frac{\alpha+\beta}{\alpha+\beta+n}\mu.$$

Example (Gamma-exponential model)

$X|\theta \sim \text{Exp}(\theta), \mu(\theta) = 1/\theta,\ f(x|\theta) = \theta e^{-\theta x}, x > 0;$
$\Theta \frown Gamma(\alpha, \beta = 1/\beta^*),$

$$\begin{aligned}
\pi(\theta) &= \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\beta\theta}\theta^{\alpha-1};\ \theta > 0; \\
L(\theta) &= \prod_{j=1}^{n} f(x_j|\theta) = \theta^n \exp\{-\theta \sum x_j\}; \\
\pi(\theta|\mathbf{x}) &= \frac{L(\theta)\pi(\theta)}{\int_0^\infty L(\theta)\pi(\theta)d\theta} \\
&= \frac{(\beta + \sum_j x_j)^{n+\alpha}}{\Gamma(n+\alpha)} \exp\{-\theta(\beta + \sum_j x_j)\}\theta^{n+\alpha-1}, \\
\pi(\theta|\mathbf{x}) &\equiv \text{Gama}(n+\alpha;\ \beta + \sum_j x_j);\ \ \mu = \mathrm{E}[X_{ij}] = \mathrm{E}[1/\theta]
\end{aligned}$$

Example (*Beta-Binomial* model cont'd)

$$\begin{aligned}
\mu &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^{+\infty} e^{-\beta\theta} \theta^{\alpha-2} d\theta = \beta \frac{\Gamma(\alpha-1)}{\Gamma(\alpha)} = \frac{\beta}{\alpha-1} \\
\mathrm{E}[1/\theta|\mathbf{x}] &= \frac{(\beta+\sum_{j=1}^n x_j)^{n+\alpha}}{\Gamma(n+\alpha)} \int_0^{+\infty} e^{-(\beta+\sum_j x_j)\theta} \theta^{n+\alpha-2} d\theta \\
&= \frac{(\beta+\sum_j x_j)\Gamma(n+\alpha-1)}{\Gamma(n+\alpha)} = \frac{\beta+\sum_j x_j}{n+\alpha-1} \\
&= \frac{n}{n+\alpha-1} \bar{x}_{\cdot j} + \frac{\alpha-1}{n+\alpha-1} \mu
\end{aligned}$$

Bühlmann's model, Empirical Bayes

Estimators are unbiased and consistent.

$$\mu = \mathrm{E}[X] = \mathrm{E}[\mathrm{E}[X|\theta]] = \mathrm{E}[\mu(\theta)].$$

$$\hat{\mu} = \bar{X} = \frac{1}{r}\sum_{i=1}^{r}\bar{X}_i = \frac{1}{nr}\sum_{i=1}^{r}\sum_{j=1}^{n}X_{ij}$$

$$\mathrm{V}[X] = \mathrm{V}[\mu(\theta)] + \mathrm{E}[\upsilon(\theta)] = a + \upsilon$$

$$\mathrm{V}[\overline{X}_i] = a + \frac{1}{n}\upsilon$$

$$\hat{\upsilon} = \frac{1}{r}\sum_{i=1}^{r}S_i'^2 = \frac{1}{r}\sum_{i=1}^{r}\sum_{j=1}^{n}\frac{\left(X_{ij} - \overline{X}_i\right)^2}{n-1}$$

$$\hat{a} = \max\left\{\frac{1}{r-1}\sum_{i=1}^{r}\left(\overline{X}_i - \bar{X}\right)^2 - \frac{1}{n}\hat{\upsilon}; 0\right\}.$$

Buhlmann-Straub's model

$$\hat{\mu} = \bar{X} = \frac{1}{m}\sum_{i=1}^{r} m_i \bar{X}_i = \frac{1}{m}\sum_{i=1}^{r}\sum_{j=1}^{n_i} m_{ij} X_{ij}$$

$$m = \sum_{i=1}^{r} m_i = \sum_{i=1}^{r}\sum_{j=1}^{n_i} m_{ij}; \qquad \hat{\mu} = \frac{\sum_{i=1}^{r} \hat{Z}_i \bar{X}_i}{\sum_{i=1}^{r} \hat{Z}_i}$$

$$\hat{\upsilon} = \frac{\sum_{i=1}^{r}\sum_{j=1}^{n_i} m_{ij}\left(X_{ij} - \overline{X}_i\right)^2}{\sum_{i=1}^{r}(n_i - 1)}$$

$$\hat{a} = \max\left\{\left(m - m^{-1}\sum_{i=1}^{r} m_i^2\right)^{-1}\left[\sum_{i=1}^{r} m_i\left(\overline{X}_i - \bar{X}\right)^2 - \hat{\upsilon}\,(r-1)\right]; 0\right\}$$

Example (A Bonus-Malus system)

Let X_j: claims in year j, $X_j \frown Poisson(\theta)$, $\mu(\theta) = v(\theta) = \theta$

$$\tilde{\theta} = \frac{n}{n + \mathrm{E}[\theta]/\mathrm{V}[\theta]}\overline{X} + \frac{\mathrm{E}[\theta]/\mathrm{V}[\theta]}{n + \mathrm{E}[\theta]/\mathrm{V}[\theta]}\mathrm{E}[\theta]$$

Data: portfolio of 106974 policies in one year (stable period):

x	0	1	2	3	4	≥ 5
n_x	96 978	9 240	704	43	9	0

- $\hat{E}[\theta] = \hat{E}[X] = \overline{X} = (1/106974)\sum_{k=0}^{4} x_k n_{x_k} = 0.1011.$
- $\hat{V}[X] = s^2 = (1/106974)\sum_{k=0}^{4} x_k n_{x_k} - \overline{x}^2 = 0.1074.$
- $\mathrm{V}[X] = \mathrm{E}[\theta] + \mathrm{V}[\theta]$. $\hat{V}[\theta] = 0.1074 - 0.1011 = 0.0063.$

Example (A Bonus-Malus system cont'd)

Risk premium/Collective premium

$$\tilde{\theta} = \frac{n}{n + 0.1011/0,0063}\overline{X} + \frac{0.1011/0.0063}{n + 0.1011/0.0063} \times 0.1011$$

$$= \left(\sum_{j=1}^{n} x_j + 16,047\,(0.1011) \right) / \,(n + 16.0476)$$

$$P^*_{n+1}(\mathbf{X}_i) = 100 \times \frac{\sum_{j=1}^{n} X_{ij} + 1.6224}{0.1011(n + 16.0476)} = 100 \times \frac{\sum_{i=1}^{n} X_{ij} + 1.6224}{0.1011\,n + 1.6224}$$

	No. of claims				
no years	0	1	2	3	4
0	100	-	-	-	-
1	94,13	152,16	210,18	268,20	326,22
2	88,92	143,72	198,53	253,34	308,14
3	84,25	136,18	188,11	240,04	291,97
4	80,05	129,39	178,73	228,06	277,40
5	76,24	123,24	170,23	217,23	264,22
6	72,79	117,65	162,51	207,38	252,24
7	69,63	112,54	155,46	198,38	241,29
8	66,73	107,86	149,00	190,13	231,26
9	64,07	103,56	143,05	182,54	222,03
10	61,61	99,58	137,56	175,53	213,50

Table: Relative premium for a *Bonus-malus* system

Example (Life group insurance)

N_{ksij}: No. people dying, with ins. capital x_k, age s, group j, year i.
$N_{ij} = \sum_{k,s} N_{ksij}$ - ...in group j year i
x_k: insured capital
q_s: mortality rate, age s, known.
$q_s\theta_j$: mortality, age s, group j (unknown)
n_{ksij}: No. people group j, capital x_k, age s, year i.
$S_{ij} = \sum_k \left(x_k \sum_s N_{ksij}\right)$: aggregate claims, group j, year i

$$\begin{aligned} N_{ksij}|\theta &\frown \text{Poisson}(n_{ksij} \times q_s \times \theta_j) \Rightarrow \\ \sum_s N_{ksij}|\theta &\sim \text{Poisson}\left(\theta_j \sum_s q_s n_{ksij}|\theta_j\right) \end{aligned}$$

Example (Life group insurance, cont'd)

$$S_{ij}|\theta = \sum_k \left(x_k \sum_s N_{ksij} \right)$$

$$S_{ij}|\theta \frown \text{CPoisson}\left(\theta_j \sum_{k,s} n_{ksij} q_s; \ f_{ij}(x) = \frac{\sum_s q_s n_{ksij}}{\sum_{k,s} q_s n_{ksij}} \right)$$

$$\mathrm{E}[S_{n+1,j}|\theta_j] = \sum_k x_k \sum_s \mathrm{E}[N_{ks(n+1)j}|\theta_j] = \theta_j \sum_{k,s} x_k q_s n_{ks(n+1)j}$$

$$P_c = \tilde{\theta}_j \sum_{k,s} x_k q_s n_{ks(n+1)j},$$

$$\tilde{\theta}_j = \frac{m_j}{m_j + \mathrm{E}[\theta_j]/\mathrm{V}[\theta_j]} \overline{X}_{\cdot j} + \frac{\mathrm{E}[\theta_j]/\mathrm{V}[\theta_j]}{m_j + \mathrm{E}[\theta_j]/\mathrm{V}[\theta_j]} \mathrm{E}[\theta_j]$$

Example (Life group insurance, cont'd)

$$\begin{aligned} \mathrm{E}[S_{n+1,j}|\theta_j] &= \sum_k x_k \sum_s \mathrm{E}[N_{ks(n+1)j}|\theta_j] = \theta_j \sum_{k,s} x_k q_s n_{ks(n+1)j} \\ P_c &= \tilde{\theta}_j \sum_{k,s} x_k q_s n_{ks(n+1)j}, \end{aligned}$$

$$\begin{aligned} \tilde{\theta}_j &= \frac{m_j}{m_j + \mathrm{E}[\theta_j]/\mathrm{V}[\theta_j]} \overline{X}_{.j} + \frac{\mathrm{E}[\theta_j]/\mathrm{V}[\theta_j]}{m_j + \mathrm{E}[\theta_j]/\mathrm{V}[\theta_j]} \mathrm{E}[\theta_j] \\ X_{ij} &= N_{ij}/m_{ij}; \quad m_{ij} = \sum_{k,s} q_s n_{ksij} \end{aligned}$$

Problem (1)

Consider a motor insurance portfolio where the population is classified into categories A, B and C, respectively, where A is Good drivers, B is Bad drivers and C is Sports drivers. The population of drivers is split as follows: 70% is in category A, 25% in B and 5% in C. For each driver in category A, there is a probability of 0.75 of having no claims in a year, a probability of 0.2 of having one claim and a probability of 0.05 of having two or more claims in a year. For each driver in category B these probabilities are 0.25, 0.4 and 0.35, respectively. For each driver in category C these probabilities are 0.3, 0.4 and 0.3, respectively.

Risk parameter representing the kind of driver is denoted by θ, which is a realization of the random variable Θ. The insurer does not know the value of that parameter. Let X be the (observable) number of claims per year for a risk taken out at random from the whole portfolio. For a given $\Theta = \theta$ yearly observations $X_1, X_2, \ldots$, make a random sample from risk X. The insurer finds crucial that the annual premium for a given risk might be adjusted by its claim record.

Consider a risk X taken out at random from the portfolio.

Calculate the mean and variance of X.

Compute the probability function of X.

For a particular risk of the portfolio we observed in the last two years $X_1 = x_1 = 0$ and $X_2 = x_2 = 2$.

For a given $\Theta = \theta$ of risk X observations, $X_1, X_2, \ldots$, are a random sample but X_1 and X_2 are not independent. Comment briefly.

Compute $Cov[X_1, X_2]$. [Note: For r.v.'s X, Y and Z,
$Cov[X, Y] = E[Cov[X, Y|Z]] + Cov[E[X|Z]; E[Y|Z]]$.]

Compute the posterior probability function of Θ given $(X_1 = 0, X_2 = 2)$.

You do not know from which risk category the above sample comes. Carry out appropriate calculations to determine from which category the sample is most likely to have come.

We need to compute a (pure) premium for the next year:

Compute the collective pure premium.

Compute the Bayes premium $E[X_3|X=(0,2)] = E(\mu(\Theta)\,|X=(0,2))$.

Compute Bühlmann's credibility premium, say, $\tilde{E}(X_3|\theta)$.

Can we talk here on Exact Credibility? Comment appropriately.

(a) Consider a risk X taken out at random from the portfolio.

i. Calculate the mean and variance of X.

ii. Compute the probability function of X.

(b) For a particular risk of the portfolio we observed in the last two years $X_1 = x_1 = 0$ and $X_2 = x_2 = 2$.

i. For a given $\Theta = \theta$ of risk X observations, $X_1, X_2, \ldots$, are a random sample but X_1 and X_2 are not independent. Comment briefly.

ii. Compute $Cov[X_1, X_2]$.
[**Note**: For r.v.'s X, Y and Z, $Cov[X, Y] = E[Cov[X, Y|Z]] + Cov[E[X|Z]; E[Y|Z]]$.]

iii. Compute the posterior probability function of Θ given $(X_1 = 0, X_2 = 2)$.

iv. You do not know from which risk category the above sample comes. Carry out appropriate calculations to determine from which category the sample is most likely to have come.

v. We need to compute a (pure) premium for the next year:

A. Compute the collective pure premium.

B. Compute the Bayes premium $E[X_3|\mathbf{X} = (0,2)] = E(\mu(\Theta)\,|\mathbf{X} = (0,2))$.

C. Compute Bühlmann's credibility premium, say, $\hat{E}(X_3|\theta)$.

D. Can we talk here on *Exact Credibility*? Comment appropriately.

Ratemaking and Experience Rating Intro

Ratemaking portfolios/groups:
Similar risks grouping in collectives of risks for ratemaking.
Tariff:Set of premia, for each risk in a (homogeneous) portfolio. A basic premium plus a system of *bonus* or *malus*.
Tariff structure: system of bonus/malus applied to a basic premium.
"**Prior" and "Posterior" ratemaking:**
First rate following given *prior* variables, then make a *posterior re-evaluation*, according to the observed accidents/claims by the risk/policy.
Bonus-malus systems, use of **GLM**'s, ..
Bonus systems are in general based on **claim counts**, not amounts. This is explained by the usual assumption of independence between **number** and **severity** of claims. The base model is Markovian.

Bonus-malus (or bonus) systems

- Common tariff in motor insurance
- ususally based on a counting variable, not the amounts
- A Markov chain model (discret time) is often used:
- Basic idea:
  - year(s) with no claim: *bonus*
  - year with 1 claims: *malus*; 2 claims: + **malus**...

Markov chain

T&K, pag.102, Ex. 2.2: A particle travels through states $\{0, 1, 2\}$ according to a Markov chain

$$P = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}; \; P^3 = \begin{bmatrix} \frac{1}{4} & \frac{3}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{1}{4} & \frac{3}{8} \\ \frac{3}{8} & \frac{3}{8} & \frac{1}{4} \end{bmatrix}; \; P^4 = \begin{bmatrix} \frac{3}{8} & \frac{5}{16} & \frac{5}{16} \\ \frac{5}{16} & \frac{3}{8} & \frac{5}{16} \\ \frac{5}{16} & \frac{5}{16} & \frac{3}{8} \end{bmatrix};$$

$$P^5 = \begin{bmatrix} \frac{5}{16} & \frac{11}{32} & \frac{11}{32} \\ \frac{11}{32} & \frac{5}{16} & \frac{11}{32} \\ \frac{11}{32} & \frac{11}{32} & \frac{5}{16} \end{bmatrix}; \; P^{10} = \begin{bmatrix} \frac{171}{512} & \frac{341}{1024} & \frac{341}{1024} \\ \frac{341}{1024} & \frac{171}{512} & \frac{341}{1024} \\ \frac{341}{1024} & \frac{341}{1024} & \frac{171}{512} \end{bmatrix}$$

Let a Markov chain with transition matrix:

$$P = \begin{bmatrix} 0 & 0.9 & 0.1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0.9 & 0 & 0.1 & 0 & 0 & 0 & 0 \\ 2 & 0.9 & 0 & 0 & 0.1 & 0 & 0 & 0 \\ 3 & 0.9 & 0 & 0 & 0 & 0.1 & 0 & 0 \\ 4 & 0.9 & 0 & 0 & 0 & 0 & 0.1 & 0 \\ 5 & 0.9 & 0 & 0 & 0 & 0 & 0 & 0.1 \\ 6 & 0.9 & 0 & 0 & 0 & 0 & 0 & 0.1 \end{bmatrix}$$

Long term:

$$P^8 = \begin{bmatrix} .9 & .09 & .009 & .0009 & .00009 & 9.0 \times 10^{-6} & 1.0 \times 10^{-6} \\ .9 & .09 & .009 & .0009 & .00009 & 9.0 \times 10^{-6} & 1.0 \times 10^{-6} \\ .9 & .09 & .009 & .0009 & .00009 & 9.0 \times 10^{-6} & 1.0 \times 10^{-6} \\ .9 & .09 & .009 & .0009 & .00009 & 9.0 \times 10^{-6} & 1.0 \times 10^{-6} \\ .9 & .09 & .009 & .0009 & .00009 & 9.0 \times 10^{-6} & 1.0 \times 10^{-6} \\ .9 & .09 & .009 & .0009 & .00009 & 9.0 \times 10^{-6} & 1.0 \times 10^{-6} \\ .9 & .09 & .009 & .0009 & .00009 & 9.0 \times 10^{-6} & 1.0 \times 10^{-6} \end{bmatrix}$$

A posterior ratemaking system, experience rating, is a *Bonus-malus* sytem if

- The rating periods are equal (1 year)
- The risks, policies, are divided into (finite) classes:

$$C_1, C_2, ..., C_s; \quad \cup_i C_i = C; \quad C_i \cap C_j = \varnothing.$$

- No transitions within the year
- Position in Class in the year n depends:
  - on position in $n-1$, and
  - the year claim counts.

Example (Centeno [2003])

A *Bonus* system in motor insurance, 3rd party liability (directly, the system is not Markovian)

- 30% discount, no claim for 2 yrs.
- 15% malus, 1 claim
- 30% malus, 2 claims
- 45% malus, 3 claims
- 100% malus, 4 claims
- > 4, case by case...

This is **not** Markovian, unless...classes are split (see later)

Composition of the B-S system:

1. A vector of *premia* (or multiplying factor, index)

$$\mathbf{b} = (b(1), b(2), ..., b(s))$$

2. Transition rules among classes, in matrix:

$$\mathbf{T} \quad = \quad [T_{ij}]\text{, each entry } T_{ij} \text{ is a set of integers...}$$

$$\mathbf{T} \quad : \quad \cup_{j=1}^{s} T_{ij} = \{0, 1, 2, ...\}\,,\ T_{ij} \cap T_{ij'} = \varnothing,\, j \neq j'$$

3. Entry class, C_{i_0} is the same for all policies.

- Symbolically, a B-M S can be written as a triplet: $\Delta = (C_{i_0}, \mathbf{T}, \mathbf{b})$.
- Bonus Class in year n: $Z_{\Delta,n}$, defined by set of rules $\mathbf{T}$ and entry class C_{i_0}.
- The system is supposed to be a Markov chain

$$\{Z_{\Delta,n}, n = 0, 1, 2, ...\}$$

- Transition probability matrix: $P_T = [p_T(i,j)]$
- Transition rules is based on claim counts, often
    - Poisson distributed (usually bad), or
    - mixed Poisson (much better), $i, j = 1, 2, ..., s$,

$$\begin{aligned} p_T(i,j) &= \Pr(Z_{\Delta,n+1} = j | Z_{\Delta,n} = i) \\ p_T^{(n)}(i,j) &= \Pr(Z_{\Delta,n} = j | Z_{\Delta,0} = i) \\ p_T^{(n)}(j) &= \Pr(Z_{\Delta,n} = j) \end{aligned}$$

Transition rules is based on claim counts, often

- Poisson distributed (usually bad), $i, j = 1, 2, ..., s$, $n = 0, 1, ...$

$$\begin{aligned} p_{T,\lambda}(i,j) &= \Pr\left(Z_{\Delta,n+1} = j | Z_{\Delta,n} = i, \Lambda = \lambda\right) \\ p^{(n)}_{T,\lambda}(i,j) &= \Pr\left(Z_{\Delta,n} = j | Z_{\Delta,0} = i, \Lambda = \lambda\right) \\ p^{(n)}_{T,\lambda}(j) &= \Pr\left(Z_{\Delta,n} = j | \Lambda = \lambda\right) \; . \end{aligned}$$

- Mixed Poisson (much better), 1st compute the conditional $p^{(n)}_{T,\lambda}(i,j)$, $i, j = 1, 2, ..., s$, then

$$\begin{aligned} p_T(i,j) &= \int_0^\infty p_{T,\lambda}(i,j) d\pi(\lambda) \\ p^{(n)}_T(i,j) &= \int_0^\infty p^{(n)}_{T,\lambda}(i,j) d\pi(\lambda) = E\left[p^{(n)}_{T,\lambda}(i,j)\right] \\ p^{(n)}_T(j) &= \int_0^\infty p^{(n)}_{T,\lambda}(j) d\pi(\lambda) = E\left[p^{(n)}_{T,\lambda}(j)\right] . \end{aligned}$$

- All B-S systems have (at least) a *bonus* class where a policy:
  - stays if keeps with no claims
  - goes, transits to, if has no claims
  - goes out, transits from (to another)
- That class is a periodic state
- If the Markov chain is irreducible, finite number of states, it will be aperiodic and stationary;
- Then, it exists a limit distribution, for a given λ

$$p_{T,\lambda}^{(\infty)}(j) = \lim_{n \uparrow \infty} p_{T,\lambda}^{(n)}(i,j).$$

If λ is considered to be the outcome of a r.v. with dist. $\pi(\lambda)$, usually

$$p_{T}^{(\infty)}(j) = \int_0^\infty p_{T,\lambda}^{(\infty)}(j) d\pi(\lambda) = E\left[p_{T,\lambda}^{(\infty)}(j)\right]$$

Remark: $p_T^{(\infty)}(j)$ is not got from the initial "mixed Poisson".

Problem (**1, cont'd**)

Consider a motor insurance portfolio where the population is classified into categories A, B *and* C, *respectively, where* A *is Good drivers,* B *is Bad drivers and* C *is Sports drivers. The population of drivers is split as follows: 70% is in category* A, *25% in* B *and 5% in* C. *For each driver in category* A, *there is a probability of 0.75 of having no claims in a year, a probability of 0.2 of having one claim and a probability of 0.05 of having two or more claims in a year. For each driver in category* B *these probabilities are 0.25, 0.4 and 0.35, respectively. For each driver in category* C *these probabilities are 0.3, 0.4 and 0.3, respectively.*

Risk parameter representing the kind of driver is denoted by θ, *which is a realization of the random variable* Θ. *The insurer does not know the value of that parameter. Let* X *be the (observable) number of claims per year for a risk taken out at random from the whole portfolio. For a given* $\Theta = \theta$ *yearly observations* $X_1, X_2, \ldots$, *make a random sample from risk* X. *The insurer finds crucial that the annual premium for a given risk might be adjusted by its claim record.*

Problem (2)

Suppose now that the insurer uses a Bonus-malus system based on the claims frequency to rate the risks of that portfolio. The system has simply three classes, numbered 1, 2 and 3 and ranked increasingly from low to higher risk. Transition rules are the following: A policy with no claims in one year goes to the previous lower class in the next year unless it is already Class 1, where it stays. In the case of a claim goes to Class 3, if it is already there no change is made.

Let $\alpha(\theta)$ *be the probability of not having any claim in one year for a policy in with risk parameter* θ*. Entry class is Class 2 and premia vector is* $b = (70, 100, 150)$.

(a) Consider a policy with risk parameter θ.

i. Write the transition rules matrix and compute the one year transition probability.

ii. Comment on the existence of the stationary distribution.

iii. Calculate the probability of a policy being ranked in Class 1 two years after entering the system.

iv. Calculate the probability function of the premium for a type A driver after two years of stay in the portfolio. Compute the average premium.

v. After some time the insurer's chief actuary concluded that for ratemaking purposes it didn't make much difference to keep categories B and C apart, and merged them into, say, B^*. For a driver in this new class, compute the probability function of the premium after one year of staying in the system (since his entry).

(b) Stationary distribution for a given θ is given by vector $\left(\alpha(\theta)^2 ; [1-\alpha(\theta)]\,\alpha(\theta) ; 1-\alpha(\theta)\right)$. Compute the probability function of the premium for a policy taken out at random from the portfolio. Calculate the average premium.

Example (Cont'd, Centeno [2003])

A *Bonus* system in motor insurance, 3rd party liability (directly, the system is not Markovian)

- 30% discount, no claim for 2 yrs.
- 15% malus, 1 claim
- 30% malus, 2 claims
- 45% malus, 3 claims
- 100% malus, 4 claims
- > 4, case by case...

This is **not** Markovian, unless...classes are split (see later)

Example (Centeno [2003]. **Class splitting**:)

- C_1 Policies with 30% *bonus*
- C_2 Policies with neither *bonus* nor *malus* for the 2nd consecutive year
- C_3 Policies with neither *bonus* nor *malus* for the 1st yr
- C_4 Policies with 15% *malus* and no claims last yr
- C_5 Policies with 15% *malus* and claims last yr
- C_6 Policies with 30% *malus* and no claims last yr
- C_7 Policies with 30% *malus* and claims last yr
- C_8 Policies with 45% *malus* and no claims last yr
- C_9 Policies with 45% *malus* and claims last yr
- C_{10} Policies with 100% *malus* and no claims last yr
- C_{11} Policies with 100% *malus* and claims last yr.

Vector of *premia* (indeces)

$$\mathbf{b} = (70, 100, 100, 115, 115, 130, 130, 145, 145, 200, 200)$$

Transition rules matrix T:

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
1	$\{0\}$				$\{1\}$		$\{2\}$		$\{3\}$		$\{4,\ldots\}$
2	$\{0\}$				$\{1\}$		$\{2\}$		$\{3\}$		$\{4,\ldots\}$
3		$\{0\}$			$\{1\}$		$\{2\}$		$\{3\}$		$\{4,\ldots\}$
4	$\{0\}$						$\{1\}$		$\{2\}$		$\{3,\ldots\}$
5				$\{0\}$			$\{1\}$		$\{2\}$		$\{3,\ldots\}$
6	$\{0\}$								$\{1\}$		$\{2,\ldots\}$
7						$\{0\}$			$\{1\}$		$\{2,\ldots\}$
8	$\{0\}$										$\{1,\ldots\}$
9								$\{0\}$			$\{1,\ldots\}$
10	$\{0\}$										$\{1,\ldots\}$
11										$\{0\}$	$\{1,\ldots\}$

If claim counts follow a Poisson(λ), $\mathbf{P}_{\Delta,\lambda}$:

	1	2	3	4	5	6	7	8	9	10	11
1	$e^{-\lambda}$				$\lambda e^{-\lambda}$		$\lambda^2 e^{-\lambda}/2$		$\lambda^3 e^{-\lambda}/6$		$1-e^{-\lambda}\sum_{i=0}^{3}\lambda^i/i!$
2	$e^{-\lambda}$				$\lambda e^{-\lambda}$		$\lambda^2 e^{-\lambda}/2$		$\lambda^3 e^{-\lambda}/6$		$1-e^{-\lambda}\sum_{i=0}^{3}\lambda^i/i!$
3		$e^{-\lambda}$			$\lambda e^{-\lambda}$		$\lambda^2 e^{-\lambda}/2$		$\lambda^3 e^{-\lambda}/6$		$1-e^{-\lambda}\sum_{i=0}^{3}\lambda^i/i!$
4	$e^{-\lambda}$						$\lambda e^{-\lambda}$		$\lambda^2 e^{-\lambda}/2$		$1-e^{-\lambda}\sum_{i=0}^{2}\lambda^i/i!$
5				$e^{-\lambda}$			$\lambda e^{-\lambda}$		$\lambda^2 e^{-\lambda}/2$		$1-e^{-\lambda}\sum_{i=0}^{2}\lambda^i/i!$
6	$e^{-\lambda}$								$\lambda e^{-\lambda}$		$1-e^{-\lambda}\sum_{i=0}^{1}\lambda^i/i!$
7	0					$e^{-\lambda}$			$\lambda e^{-\lambda}$		$1-e^{-\lambda}\sum_{i=0}^{1}\lambda^i/i!$
8	$e^{-\lambda}$										$1-e^{-\lambda}$
9								$e^{-\lambda}$			$1-e^{-\lambda}$
10	$e^{-\lambda}$										$1-e^{-\lambda}$
11										$e^{-\lambda}$	$1-e^{-\lambda}$

The Markov chain is not irreducible. You cannot go to Class/state 3. Class of states $\{C_2, C_3\}$ is transient. Class, $\{C_1, C_4, C_5, C_6, C_7, C_{8,} C_9, C_{10}, C_{11}\}$ is a class of positive recurrent aperiodic states. Re-order states

$$\mathbf{P}_{\Delta,\lambda} = \begin{bmatrix} \mathbf{P}_{1,(\Delta,\lambda)} & \mathbf{P}_{3,(\Delta,\lambda)} \\ \mathbf{0} & \mathbf{P}_{2,\Delta,\lambda} \end{bmatrix}$$

$\mathbf{P}_{1,\Delta,\lambda}$: Transition Prob'ty matrix between $\{C_2, C_3\}$
$\mathbf{P}_{3,\Delta,\lambda}$: Transition Prob'ty matrix among $\{C_2, C_3\}$ e $\{C_1, C_4, C_5, C_6, C_7, C_{8,} C_9, C_{10}, C_{11}\}$
$\mathbf{P}_{2,\Delta,\lambda}$: Transition Prob'ty matrix among $\{C_1, C_4, C_5, C_6, C_7, C_{8,} C_9, C_{10}, C_{11}\}$.

$$\mathbf{P}^2_{\Delta,\lambda} = \begin{bmatrix} \mathbf{0} & \mathbf{P}_{1,(\Delta,\lambda)}\mathbf{P}_{3,(\Delta,\lambda)} + \mathbf{P}_{3,(\Delta,\lambda)}\mathbf{P}_{2,(\Delta,\lambda)} \\ \mathbf{0} & \mathbf{P}^2_{2,(\Delta,\lambda)} \end{bmatrix};$$

$$\mathbf{P}^2_{1,\Delta,\lambda} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix}^2$$

$$\mathbf{P}^n_{\Delta,\lambda} = \begin{bmatrix} \mathbf{0} & \left(\mathbf{P}_{1,(\Delta,\lambda)}\mathbf{P}_{3,(\Delta,\lambda)} + \mathbf{P}_{3,(\Delta,\lambda)}\mathbf{P}_{2,(\Delta,\lambda)}\right)\mathbf{P}^{n-2}_{2,(\Delta,\lambda)} \\ \mathbf{0} & \mathbf{P}^n_{2,(\Delta,\lambda)} \end{bmatrix}$$

In our example

$$
\mathrm{P}_{2,\Delta,\lambda} = \begin{bmatrix}
e^{-\lambda} & & \lambda e^{-\lambda} & & \lambda^2 e^{-\lambda}/2 & & \lambda^3 e^{-\lambda}/6 & & 1 - e^{-\lambda}\sum_{i=0}^{3} \lambda^i / i! \\
e^{-\lambda} & & & & \lambda e^{-\lambda} & & \lambda^2 e^{-\lambda}/2 & & 1 - e^{-\lambda}\sum_{i=0}^{2} \lambda^i / i! \\
 & e^{-\lambda} & & & \lambda e^{-\lambda} & & \lambda^2 e^{-\lambda}/2 & & 1 - e^{-\lambda}\sum_{i=0}^{2} \lambda^i / i! \\
e^{-\lambda} & & & & & & \lambda e^{-\lambda} & & 1 - e^{-\lambda}\sum_{i=0}^{1} \lambda^i / i! \\
0 & & & e^{-\lambda} & & & \lambda e^{-\lambda} & & 1 - e^{-\lambda}\sum_{i=0}^{1} \lambda^i / i! \\
e^{-\lambda} & & & & & & & & 1 - e^{-\lambda} \\
 & & & & & e^{-\lambda} & & & 1 - e^{-\lambda} \\
e^{-\lambda} & & & & & & & & 1 - e^{-\lambda} \\
 & & & & & & & e^{-\lambda} & 1 - e^{-\lambda}
\end{bmatrix}
$$

$$\lambda = 0.1$$

Recover

$$\mathbf{P}^n_{\Delta,\lambda} = \begin{bmatrix} \mathbf{0} & \left(\mathbf{P}_{1,(\Delta,\lambda)}\mathbf{P}_{3,(\Delta,\lambda)} + \mathbf{P}_{3,(\Delta,\lambda)}\mathbf{P}_{2,(\Delta,\lambda)}\right)\mathbf{P}^{n-2}_{2,(\Delta,\lambda)} \\ \mathbf{0} & \mathbf{P}^n_{2,(\Delta,\lambda)} \end{bmatrix}$$

Calculatin the limit

$$\begin{aligned} \lim_{n\to\infty} \mathbf{P}^n_{\Delta,\lambda} &= \begin{bmatrix} \mathbf{0} & \left(\mathbf{P}_{1,(\Delta,\lambda)}\mathbf{P}_{3,(\Delta,\lambda)} + \mathbf{P}_{3,(\Delta,\lambda)}\mathbf{P}_{2,(\Delta,\lambda)}\right)\lim \mathbf{P}^{n-2}_{2,(\Delta,\lambda)} \\ \mathbf{0} & \lim_{n\to\infty} \mathbf{P}^n_{2,(\Delta,\lambda)} \end{bmatrix} \\ \mathbf{P}^\infty_{2,(\Delta,\lambda)} &= \lim_{n\to\infty} \mathbf{P}^{n-2}_{2,(\Delta,\lambda)} \text{ and } \mathbf{P}^\infty_{2,(\Delta,\lambda)} = \mathbf{P}^\infty_{2,(\Delta,\lambda)}\mathbf{P}_{2,(\Delta,\lambda)} \\ \text{or } \mathbf{0} &= \mathbf{P}^\infty_2 \left(\mathbf{I} - \mathbf{P}_2\right) \end{aligned}$$

$\mathbf{P}^n_{\Delta,\lambda}$ tends for a matrix with all lines equal, of the form

$$\mathbf{P}^n_{\Delta,\lambda} \to \left[\mathbf{0} \mid \mathbf{P}^\infty_{2,(\Delta,\lambda)}\right]$$

Example with $\lambda = 0.1$, we get

$$\begin{pmatrix} 0.81873 & 0.067032 & 0.074082 & 0.014905 & 0.016473 & 0.0032584 \\ & & 0.0036011 & 91126 \times 10^{-4} & 10071 \times 10^{-3} & \end{pmatrix}$$

- Lemaire's (1995):
  - *Relative Stationary Average Level* (RSAL):

$$
\begin{aligned}
RSAL &= \frac{SAP - mP}{MP - mP} \\
SAP &= \sum_{j=1}^{s} b(j) p_T^{(\infty)}(j)
\end{aligned}
$$

    SAP: Stationary Average Premium, mP: minimum Premium, MP: Max Premium

  - Premium variation coefficient (VC):

$$
\begin{aligned}
RSAL &= SDP / SAP \\
SDP &= \sqrt{\sum_{j=1}^{s} b(j)^2 p_T^{(\infty)}(j) - SAP^2}
\end{aligned}
$$

- Elasticity of the average premium (Response to changes in frequency mean)

$$\eta(\lambda) = \frac{\frac{dSAP(\lambda)}{SAP}}{\frac{d\lambda}{\lambda}} = \frac{d\ln SAP(\lambda)}{d\ln\lambda}$$

  If

$$\begin{aligned} \lambda &\to \infty \Rightarrow SAP(\lambda) \to \max\{b(j)\} < \infty; \\ \lambda &\to \infty \Rightarrow \eta(\lambda) \to 0; \qquad \lambda \to 0 \Rightarrow \eta(\lambda) \to 0. \end{aligned}$$

- Lemaire's (1985) *Transient Elasticity* (1st step analysis)

$$V_\lambda(j) = b(j) + \beta_j \sum_{k=1}^{s} p_{T,\lambda}(j,k) V_\lambda(k), \;\; j = 1, ..., s$$

  - $V_\lambda(j)$: Expected present value to be paid by popli from C_j;
  - β_j (< 1): Discount rate.

- Lemaire's (1985) *Transient Elasticity* (1st step analysis)

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  - $V_\lambda(j)$: Expected present value to be paid by popli from C_j;
  - β_j (< 1): Discount rate.

The system has a unique solution and elasticity comes:

$$\mu_\lambda(j) = \frac{dV_\lambda(j)/V_\lambda(j)}{d\lambda/\lambda}$$

$$\mu(j) = \int_0^\infty \mu_\lambda(j) d\pi(\lambda)$$

"Bonus hunger"

- Due to "claims frequency system"
- (Some?) Small accidents aren't reported;
  - It changes: the reported frequency and amonts dist's;
  - Decreases insurer's management costs;
  - "No-report" decision depends:
    - solely **on insuree**, and
    - his bonus class C_j;
- Let x_j: Retention level (works like a "Franchise" not a "Deductible");
- It's possible to find an optimal retention point: x_j^* (under some assumptions).

Hypothesis:

- (Unreal) Insuree knows single amount distr. $F_X(\cdot)$, and x_j;
- $N \frown Poisson(\lambda)$; Single amount $X_i \frown F_X(\cdot)$;

  Let N^*: no. of accidents declared in C_j:

$$N^* = \sum_{i=0}^{N} Y_i, \quad Y_0 \equiv 0$$

$$Y_i \frown binomial(1; p); \qquad p = \Pr[X_i > x_j] = \bar{F}_X(x_j).$$

Then

$$N^* \frown CPoisson(\lambda, F_y) \equiv Poisson(\lambda \bar{F}_X(x_j))$$

Let D : Cost of unreported claim.

$$D(x_j) = XI_{\{X \leq x_j\}}$$

Hypothesis (cont'd)

- Mean cost of unreported accidents:

$$E\left[D(x_j)\right] = 0 \times \lambda \bar{F}_X(x_j) + \lambda F_X(x_j)$$

- and payments are made in mid-year:

$$V_{\lambda,\mathbf{x}}(j) = b(j) + \beta^{1/2} E\left[D(x_j)\right] + \beta \sum_{k=1}^{s} p_{T,\lambda,x_j}(j,k) V_{\lambda,\mathbf{x}}(k), \;\; j = 1, ..$$

Matrix form equation:

$$\begin{aligned} \mathbf{V}_{\lambda,\mathbf{x}} &= \mathbf{b}(\mathbf{x}) + \beta \mathbf{P}_{T,\lambda,\mathbf{x}}(j,k) \mathbf{V}_{\lambda,\mathbf{x}} \\ \mathbf{V}_{\lambda,\mathbf{x}} &= \left(\mathbf{I} - \beta \mathbf{P}_{T,\lambda,\mathbf{x}}\right)^{-1} \mathbf{b}(\mathbf{x}) \\ \mathbf{b}(\mathbf{x})' &= \left(..., b(j) + \beta^{1/2} E\left[D(x_j)\right], ...\right). \end{aligned}$$

Under those conditions it's possible to find optimums x_j^*, see Centeno(2003, pp 181-184), and for algorithms.

- Norberg's (1976) model. Efficiency measure, $Q_n(\Delta)$:

$$Q_n(\Delta) = E\left(\left[b_n(Z_{\Delta,n}) - E\left(S_n|\lambda\right)\right]^2\right)$$

$$\begin{aligned}
\text{Bonus class in } n &: \quad Z_{\Delta,n}, \quad n = 0, 1, 2, \ldots \\
S_n &: \quad \text{Aggregate claims of policy in } n \\
E\left(S_n|\lambda\right) &: \quad \text{Risk premium, unknown.}
\end{aligned}$$

$$\begin{aligned}
Q_n(\Delta) &= E\left(\left[b_n(Z_{\Delta,n}) - E\left(S_n|\lambda\right)\right]^2\right) \qquad \text{(Like in credibility)} \\
&= E\left[E\left(\left[b_n(Z_{\Delta,n}) - E\left(S_n|\lambda\right)\right]^2\right)|Z_{\Delta,n}\right] \\
&= E\left[V\left(\left[b_n(Z_{\Delta,n}) - E\left(S_n|\lambda\right)\right]^2\right)|Z_{\Delta,n}\right] \\
&\quad + E\left[\left(E\left[b_n(Z_{\Delta,n}) - E(E\left(S_n|\lambda\right)\right]|Z_{\Delta,n})\right)^2\right] \\
&\geq E\left[V\left(\left[b_n(Z_{\Delta,n}) - E\left(S_n|\lambda\right)\right]^2\right)|Z_{\Delta,n}\right]
\end{aligned}$$

- Statistical modelling
  - Model the pure premium
  - Model the Conditional Expected Value:

$$\begin{aligned} E(Y|x_1, x_2, ..., x_p) &= h(x_1, x_2, ..., x_p, \beta_1, \beta_2, ..., \beta_p) \\ Y &= h(x_1, x_2, ..., x_p, \beta_1, \beta_2, ..., \beta_p) + \varepsilon \end{aligned}$$

Y: endogenous variable, x_i: factor, exogenous, β_j: parameter

- Different sorts of variables: **Nominal** (binary: sex, good/bad risk), **ordinal**/**Categorical** (ranks: age, power groups), **discrete** (age, experience yrs, claim counts...), **continuous** (income, cliam amounts)
- Data, Information must be (always) reliable, as simple as possible, clean, neat...
- Y: Pure premium, Factors: risk factors influencing:
  - E.g motor insurance: kms, traffic, driver's ability, power, vehicle type, driver's experience, geographical factors...

Deal with the experts about the factors influencing, gather information, data (manageable data)
In motor insurance we can consider

- Past accident record
- kms driven
- Car owner (company/private)
- Use (business or private)
- Vehicle value
- Power (cm^3)
- Weight
- Driver's age
- Driving region (usual, city/countryside...)
- Multiple driver's?
- Vehicle age

- Years fo driver's expereince
- Car brand and/or model
- Sex
- Sort of insurance (third party, own damages)
- Driver's profession
- etc,...
- ....
- Built classes of factors. Often Class aggregation is needed
- Often we have many binary or rank variables, qualitative data

If dependent variable Y is:

- Binary: Model a *Logit* or *Probit*
- Countig data: *Poisson* model. Ex: Number of claims in a Bonus system
- Continuous data: *Gamma* model. Ex: Amount of claims
- ...

Let S be Aggregate claims in one year. Then $E(S) = E(N)E(X)$, is the **pure premium** (N is annual number of claims and X is amount of each claim). We can consider modeling the two expectations separately.

In a portfolio we can consider different level factors influencing each (conditional) expectation, building a tariff, such that:

$$E(Y|x_1, x_2, ..., x_p) = h(x_1, x_2, ..., x_p, \beta_1, \beta_2, ..., \beta_p)$$

Specifying $h(x_1, x_2, ..., x_p, \beta_1, \beta_2, ..., \beta_p)$ may not be an easy task, where the $x_1, x_2, ..., x_p$ are the factors.

A tariff analysis is based on insurer's own data.
Steps:

- Postulate a distribution of Y according to its nature, as well as the factors $(x_1, x_2, \ldots, x_p)$;
- Based on a sample for Y and $(x_1, x_2, \ldots, x_p)$ choose the *best* $h(.)$ and estimate $(\beta_1, \beta_2, \ldots, \beta_p)$;
- Hypothesis testing, for Y and $(x_1, x_2, \ldots, x_p)$.

We should consider:

- Existing information in the company;
- Used variables in other, previous, studies;
- Market used variables;
- Legal limitations.

Data:

- Must be reliable, objective;
- Number of variables must be adequate, no too long or too short;
- All information must cover an homogeneous period. Not too long periods, e.g.

Models:

- Additive models. ANOVA;
- Mutliplicative models, GLM, e.g. two factors:

$$\mu_{ij} = \gamma_0 \gamma_{1i} \gamma_{2j}$$