Master in Actuarial Science

Models in Finance
07-01-2014

Solutions

1. (a) By Itô's lemma (or Itô's formula) applied to $g(t,x)$ (it is a $C^{1,2}$ function):

$$\begin{aligned}
dg(t,S_t) &= \frac{\partial g}{\partial t}(t,S_t)dt + \frac{\partial g}{\partial x}(t,S_t)dS_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,S_t)\,(dS_t)^2 \\
&= \left[\frac{\partial g}{\partial t}(t,S_t) + \alpha(t,S_t)\frac{\partial g}{\partial x}(t,S_t) + \frac{1}{2}\left(\sigma(t,S_t)\right)^2 \frac{\partial^2 g}{\partial x^2}(t,S_t)\right] dt \\
&+ \sigma(t,S_t)\frac{\partial g}{\partial x}(t,S_t)dB_t \\
&= 0 + \sigma(t,S_t)\frac{\partial g}{\partial x}(t,S_t)dB_t
\end{aligned}$$

where we have used $(dB_t)^2 = dt$.

(b) We have

$$dS_t = 0.05 S_t dt + 0.15 S_t dB_t,$$

which is the SDE of a geometric Brownian motion with $\alpha = 0.05$ and $\sigma = 0.15$. The solution is (it can be obtained by applying the Itô formula to $f(x) = \log(1/x)$)

$$\begin{aligned}
S_t &= S_0 \exp\left[\left(\alpha - \frac{1}{2}\sigma^2\right)t + \sigma B_t\right] \\
&= S_0 \exp\left[\left(0.05 - \frac{1}{2}(0.15)^2\right)t + 0.15 B_t\right]
\end{aligned}$$

Therefore

$$S_t = S_0 \exp\left[0.03875t + 0.15B_t\right].$$

Since $B_t \sim N(0;t)$, then $\log(S_t) \sim N(\log(S_0) + 0.03875t; 0.0225t)$.

$$\begin{aligned}
P\left(\frac{S_5}{S_0} > 1.2\right) &= P\left(\exp\left[0.03875 \times 5 + 0.15B_5\right] > 1.2\right) \\
&= 1 - P\left(Z \le \log(1.2)\right).
\end{aligned}$$

where $Z = 0.19375 + 0.15B_5 \sim N\left(0.19375; 0.1125\right)$.

Therefore: $P\left(\frac{S_5}{S_0} > 1.2\right) = 1 - 0.4864 = 0.5136$.

(a) $QMU$ is the long-run mean for the force of inflation.

$QA$ is the autoregressive parameter: its value should be such that $0 < QA < 1$ in order to have a mean reverting inflation.

$QSD.QZT$ represents the random component of the process or "the random shock to the system" part of the equation. $QSD$ dictates the typical size of this "shock"

(b) $I(t-1) = 0.06$

$$\begin{aligned} I(t) &= 0.04 + 0.5 \times [0.06 - 0.04] + 0.045QZ(t) \\ &= 0.05 + 0.045QZ(t), \end{aligned}$$

where $QZ(t) \sim N(0,1)$. In order to obtain the 90% confidence interval, from the percentage points table for the standard normal distribution, we have that for the upper level:

$$I(t)_{\max} = 0.05 + 0.045 \times 1.6449 = 0.1240,$$

and for the lower level

$$I(t)_{\min} = 0.05 - 0.045 \times 1.6449 = -0.0240.$$

and the 90% confidence interval is $\left[-0.0240, 0.1240\right]$.

(c) Four examples of financial/economical variables that should be modelled by auto-regressive models, with a mean-reversion effect: interest rates, dividend yields, rate of inflation, annual rate of growth in dividends.

A random walk process can be expected to grow arbitrarily large with time. If share prices follow a random walk, with positive drift, then those share prices would be expected to tend to infinity for large time horizons. However, there are many quantities which should not behave like this. For example, we do not expect interest rates to jump off to infinity, or to collapse to zero. Instead, we would expect some mean reverting force to pull interest rates back to some normal range. In the same way, while dividend yields can change substantially over time, we would expect them, over the long run, to form some stationary distribution, and not run off to infinity. Similar considerations apply to the annual rate of growth in prices or in dividends. In each case, these quantities are not independent from one year to the next; times of high interest rates or high inflation tend to bunch together i.e. the models are autoregressive.

(a) At time $t$, consider the portfolio: one European put + Share $S_t$ and a cash account with value $Ke^{-r(T-t)}$. At time $T$, the portfolio value is $0 + S_T = S_T > K$ if $S_T > K$. If $S_T < K$ then the payoff from portfolio is $K - S_T + S_T = K$. The cash account at time $T$ has a value of $K$. Therefore the portfolio payoff $\geq K$ $\implies p_t + S_t \geq Ke^{-r(T-t)}$ and we have the lower bound for the price of European put:

$$p_t \geq Ke^{-r(T-t)} - S_t.$$

(b) By the put-call parity:

$$c_t + Ke^{-r(T-t)} = p_t + S_t.$$

Therefore:

$$\begin{aligned} Ke^{-r(T-t)} &= p_t + S_t - c_t \\ (T-t) &= -\frac{1}{r}\log\left(\frac{p_t + S_t - c_t}{K}\right) \end{aligned}$$

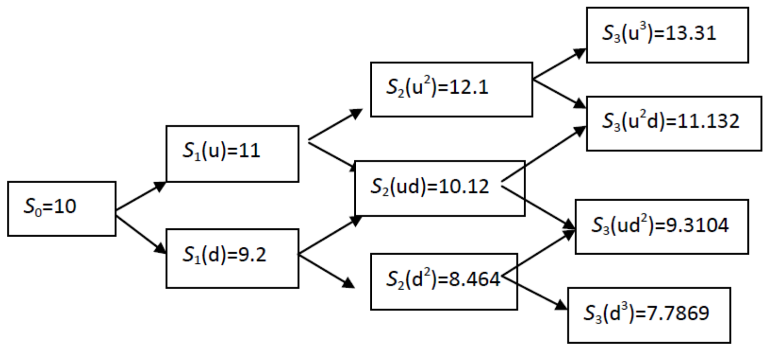
and

$$\begin{aligned} (T-t) &= -\frac{1}{0.04}\log\left(\frac{0.9 + 16.5 - 1.2}{17}\right) \\ &= 1.2051 \end{aligned}$$

and the time to expiry is $T - t = 1.2051$ years.

(c) It is never optimal to exercise an american call on a non-dividend paying share early because if we exercise early, the payoff is $S_t - K$, but if we do not exercise, the value of the American call must be at least that of the European call, i.e., by the lower bound for an European call option, $C_t \geq S_t - Ke^{-r(T-t)} > S_t - K$. So, we would receive more by selling the option than by exercising it.

(a) $\frac{S_{t+1}}{S_t} = 1.10$ or $\frac{S_{t+1}}{S_t} = 0.92$. Therefore $u = 1.10$ and $d = 0.92$. $e^r = e^{0.05} = 1.0513$ and we have $d < e^r < u$ and therefore the model is arbitrage free.
Binomial tree:




(b) The risk neutral probability of an up-movement is

$$q = \frac{e^r - d}{u - d} = \frac{e^{0.05} - 0.92}{1.1 - 0.92} = 0.7293.$$

Payoff of the derivative: $C_3 = \max\left\{(S_3)^2 - K, 0\right\}$ with $K = 130$.
Using the usual backward procedure:
$C_3\left(u^3\right) = \max\left\{\left(S_0 u^3\right)^2 - 130, 0\right\} = 47.1561$, $C_3(u^2 d) = \max\left\{\left(S_0 u^2 d\right)^2 - 130, 0\right\} = 0$, $C_3\left(d^2 u\right) = \max\left\{\left(S_0 d^2 u\right)^2 - 130, 0\right\} = 0$ and $C_3\left(d^3\right) = \max\left\{\left(S_0 d^3\right)^2 - 130, 0\right\} = 0$.
At time 2: $C_2\left(u^2\right) = \exp(-r)\left[q C_3\left(u^3\right) + (1-q) C_3\left(u^2 d\right)\right] = 32.7137$, $C_2(ud) = 0$, $C_2\left(d^2\right) = 0$.
At time 1: $C_1(u) = \exp(-r)\left[q C_2\left(u^2\right) + (1-q) C_2(ud)\right] = 22.6946$, $C_1(d) = 0$
At time 0, the price is $C_0 = \exp(-r)\left[q C_1(u) + (1-q) C_1(d)\right] = 15.7439$.
Or, we can calculate by $C_0 = e^{-3r} q^3 C_3\left(u^3\right) = 15.7439$.

(a) 1.Establish the equivalent martingale measure $Q$ under which $D_t = e^{-rt} S_t$ is a martingale.
2. Propose $V_t = e^{-r(T-t)} E_Q\left[X \mid \mathcal{F}_t\right]$ as the "fair" price of the derivative.
3. Show that $E_t = e^{-rt} V_t = e^{-rT} E_Q\left[X \mid \mathcal{F}_t\right]$ is a martingale under $Q$.
4. Use the Martingale representation theorem to construct a hedging strategy (portfolio) $(\phi_t, \psi_t)$.

5. Show that the hedging strategy $(\phi_t, \psi_t)$ replicates the derivative payoff at maturity and therefore $V_t$ is the fair price of the derivative at time $t$.

(b) A portfolio with zero delta means that: (number of put options)$\times$(*delta* of one put) + number of shares ($N$) = 0, since the delta of a share is 1. Therefore: $20000 \times (-0.20) + N = 0$ and therefore $N = 4000$ shares.

(c) The option price is given by:

$$p_t = Ke^{-r(T-t)}\Phi(-d_2) - S_t e^{-q(T-t)}\Phi(-d_1).$$

with:

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} = \frac{\ln\left(\frac{45}{50}\right) + \left(0.04 + \frac{0.15^2}{2}\right) \times 0.75}{0.15\sqrt{0.75}},$$

$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

Therefore, $d_1 = -0.51517$, $d_2 = -0.64507$ and

$$\begin{aligned} p_t &= 50e^{-0.06\times0.75}\Phi(0.64507) - 45e^{-0.02\times0.75}\Phi(0.51517) \\ &= 4.5102. \end{aligned}$$

(d) Desirable characteristics of term structure models:

1. The model should be arbitrage free.

2. Interest rates should be positive.

3. $r(t)$ and other interest rates should be mean-reverting.

4. Computational efficiency: we aim for models which either give rise to simple formulae for bond and option prices or which make it straightforward to compute prices using numerical techniques.

5. The model should reproduce realistic dynamics for the interest rates and bond prices.

6. The model, with appropriate parameter estimates, should fit historical interest-rate data.

7. The model should be easily and accurately calibrated to current market data.

8. The model should be flexible enough to cope properly with a range of derivative contracts.

(e) Spot rate: $R(t,T) = \frac{-1}{T-t} \log B(t,T)$, instantaneous forward rate: $f(t,T) = -\frac{\partial}{\partial T} \log B(t,T)$.

Therefore:

$$
\begin{aligned}
R(t,T) &= r(t) - \frac{\sigma^2}{6} (T-t)^2 , \\
f(t,T) &= -\frac{\partial}{\partial T} \left[ -(T-t) r(t) + \frac{\sigma^2}{6} (T-t)^3 \right] \\
&= r(t) - \frac{\sigma^2}{2} (T-t)^2 .
\end{aligned}
$$

For the market price of risk, $\gamma(t,T) = \frac{m(t,T) - r(t)}{S(t,T)}$, where $dB(t,T) = B(t,T) \left[ m(t,T) dt + S(t,T) dZ_t \right]$.

By Itô's formula, we have that

$$
\begin{aligned}
dB(t,T) &= \frac{\partial B(t,T)}{\partial t} dt + \frac{\partial B(t,T)}{\partial r_t} dr(t) + \frac{1}{2} \frac{\partial^2 B(t,T)}{\partial r_t^2} (dr(t))^2 \\
&= B(t,T) \left[ (r(t) - \alpha(T-t) r(t)) \, dt - \sigma(T-t) dZ_t \right] .
\end{aligned}
$$

Therefore, $S(t,T) = -\sigma(T-t)$, $m(t,T) = r(t) - \alpha(T-t) r(t)$ and

$$
\gamma(t,T) = \frac{r(t) - \alpha(T-t) r(t) - r(t)}{-\sigma(T-t)} = \frac{\alpha}{\sigma} r(t).
$$