Models in Finance - Class 1

Master in Actuarial Science

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ISEG

Programme - Models in Finance

I – Stochastic Calculus
1. Brownian motion and martingales
2. The Itô integral
3. Itô's Formula
4. Stochastic Differential Equations
5. The Girsanov Theorem
6. Stochastic models of security prices

II – Valuation of derivative securities

7. Introduction to the valuation of derivative securities
8. The Binomial model
9. The Black-Scholes model

III – Term structure and credit risk models

10. Models for the term structure of interest rate
11. Credit risk models

Main Bibliography

• Björk, Tomas (2004), Arbitrage Theory in Continuous Time, second edition, Oxford University Press.
• Hull, J. (2008) Options, futures and other derivatives, 7th ed., Prentice Hall.
• Mikosch, T. (1999) , Elementary Stochastic Calculus With Finance in View, World Scientific Publishing Company.
• Nualart, D. (2008), Stochastic Calculus, Lecture Notes on Stochastic Calculus, http://www.math.ku.edu/~nualart/StochasticCalculus.pdf
• Oksendal, B. (2003), Stochastic Differential Equations: An Introduction with Applications, 6th edition, Springer.
• Core Reading for the 2014 examinations, Subject CT8, The Actuarial Profession, Institute and Faculty of Actuaries, 2013.

Assessment: The final grade is awarded on the basis of a written exam.

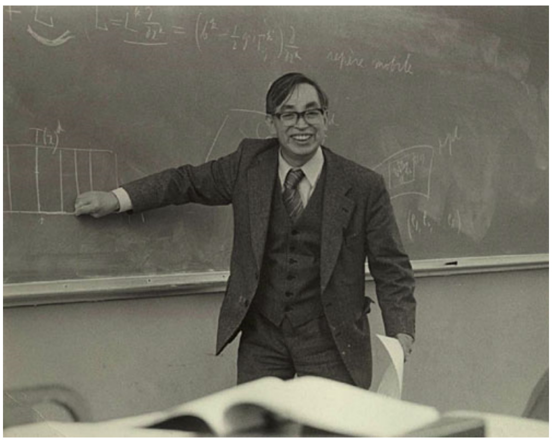
Stochastic calculus

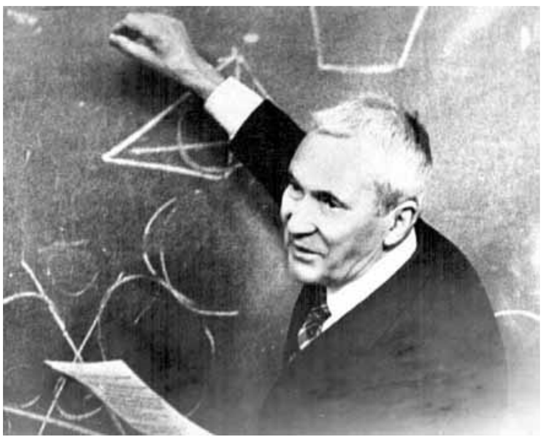
- What is stochastic calculus?
- It is an integral (and differential) calculus with respect to certain stochastic processes (for example: Brownian motion).
- It allows to define integrals (and "derivatives") of stochastic processes where the "integrating function" is also a stochastic process.
- It allows to define and solve stochastic differential equations (where there is a random factor).
- Most important stochastic process for stochastic calculus and financial applications: Brownian motion
- Main financial applications: Pricing and hedging of financial derivatives, the Black-Scholes model, interest rate models and credit risk modelling.

History of stochastic calculus and financial applications

See Robert Jarrow and Philip Protter: "A short History of Stochastic Integration and Mathematical Finance" in http://people.orie.cornell.edu/~protter/WebPapers/historypaper6.pdf

- T. N. Thiele
- Louis Bachelier
- Albert Einstein
- Norbert Wiener
- Kolmogorov
- Vincent Doeblin
- Kiyosi Itô
- Doob
- P. A. Meyer
- Black, Scholes and Merton
- etc...





Stochastic Process - Definition

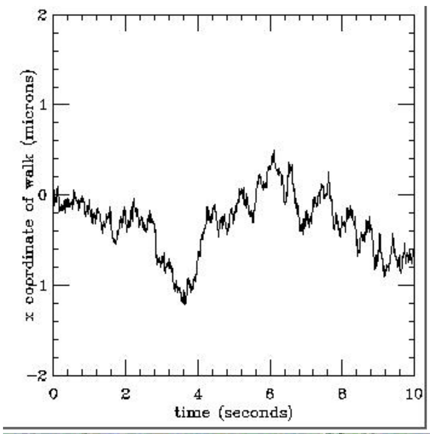
- What is a stochastic process?

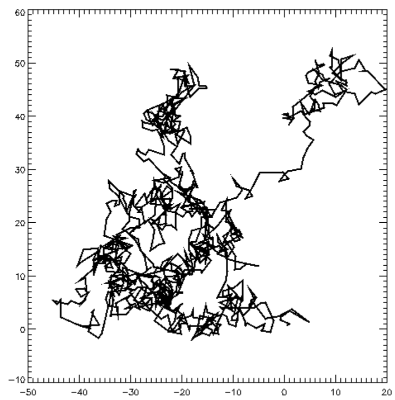
Definition

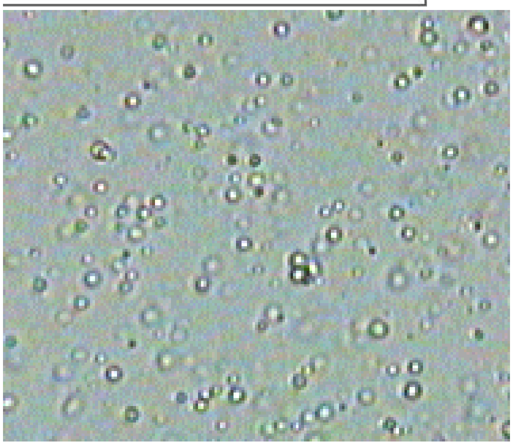
A stochastic process is a family of random variables $\{X_t, t \in T\}$ defined on a probability space $(\Omega, \mathcal{F}, P)$. T: set where the "time" parameter t is defined. If $T = \mathbb{N}$, we say that the process is a discrete time process; if $T = [a, b] \subset \mathbb{R}$ or if $T = \mathbb{R}$, we say it is a continuous time process.

- $\{X_t, t \in T\} = \{X_t(\omega), \omega \in \Omega,\ t \in T\}$
- X_t:state or position of the process at time t.
- The space of the states **S** (space where the random variables take values) is usually $\mathbb{R}$ (continuous state space) or $\mathbb{N}$ (discrete state space).
- For each fixed ω ($\omega \in \Omega$), the mapping $t \to X_t(\omega)$ or $X_\cdot(\omega)$ is called a realization, trajectory or sample path of the process.

Brownian motion







Brownian motion - History

- Robert Brown: 19th century botanist who first observed the physical motion of grains of polen suspended in water under a microscope - this type of motion is now known as "Brownian motion".
- Louis Bachelier : Théorie de la spéculation (1900): thesis on the application of Brownian motion to model financial assets evolution.
- Albert Einstein: in one of his 1905 papers, used Brownian motion as a tool to indirectly confirm the existence of atoms and molecules.

Standard Brownian motion - Definition

Definition

Standard Brownian motion (also called Wiener Process) is a stochastic process $B = \{B_t; t \geq 0\}$ with state space $\mathbf{S} = \mathbb{R}$ and satisfying the following properties:

1. $B_0 = 0$.
2. B has independent increments (i.e. $B_t - B_s$ is independent of $\{B_u, u \leq s\}$ whenever $s < t$)
3. B has stationary increments (i.e., the distribution of $B_t - B_s$ depends only on $t - s$).
4. B has Gaussian increments (i.e., the distribution of $B_t - B_s$ is the normal distribution $N(0, t - s)$.)
5. B has continuous sample paths (i.e. for each fixed ω ($\omega \in \Omega$), the mapping $t \to X_t(\omega)$ is continuous).

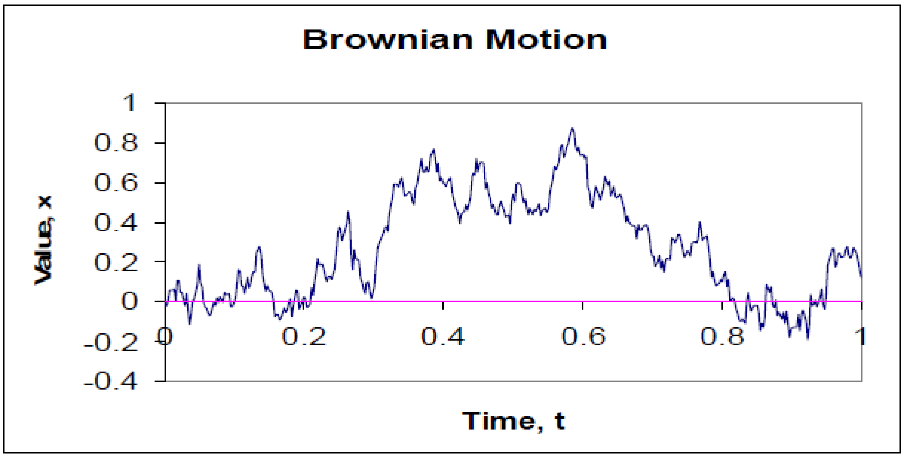


Figure 7.1: a typical trajectory of Brownian motion

Brownian motion

- Some authors (including the authors of CT8 core reading) consider that the term Brownian motion refers to a process $W = \{W_t; t \geq 0\}$ which satisfies conditions 2,3 e 5 of the previous definition of Standard Brownian motion (sBm) and also condition 4': the distribution of $W_t - W_s$ is $N(\mu (t-s), \sigma^2 (t-s))$.
- μ:drift coefficient; σ:diffusion coefficient.
- A Brownian motion (Bm) W with drift μ and diffusion coefficient σ can be constructed from a standard Brownian motion (sBm) B by:

$$W_t = W_0 + \mu t + \sigma B_t.$$

- Exercise: Prove the previous statement, i.e., prove that if B is a sBm, then $W_t = W_0 + \mu t + \sigma B_t$ is a Bm with drift μ and diffusion coeffic. σ.

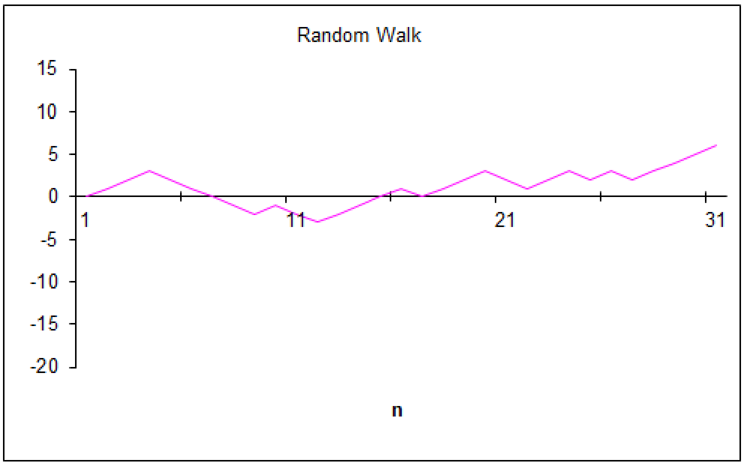
- Note: It can be difficult to prove that the conditions on the definition of sBM are compatible. However, using the Kolmogorov extension theorem (see Oksendal's book) one can show that there exists a stochastic process which satisfies all the conditions of the definition of sBm.
- Condition (4) or condition (5) can be dropped from the definition of sBm or Bm, since each of these properties can be shown to be a consequence of the other properties.
- Brownian motion is the only process with stationary independent increments and continuous sample paths.

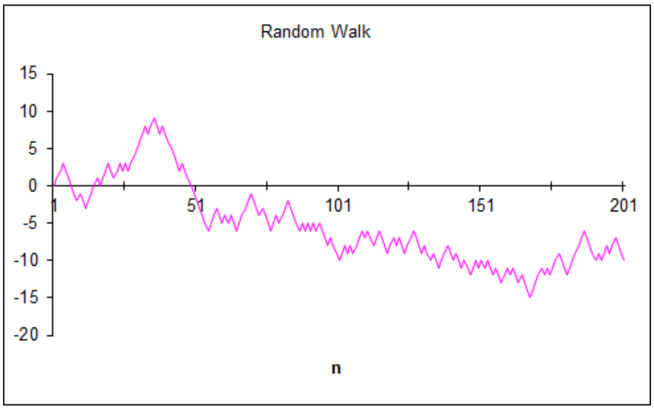
Brownian motion and random walk

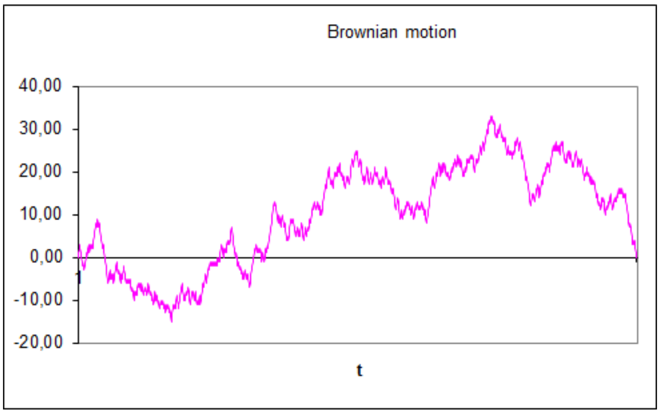
- Simple symmetric random walk: discrete time process defined by

$$X_n = \sum_{i=1}^{n} Z_i, \text{ where } Z_i = \begin{cases} +1 & \text{with probability } \frac{1}{2}. \\ -1 & \text{with probability } \frac{1}{2}. \end{cases}$$

- If we reduce the step size progessively from 1 unit until it is infinitesimal (and rescale the values of X) then the simple symmetric random walk tends to a Brownian motion.







Properties of Brownian motion

- In order to prove some properties, we can use the following decomposition (for $s < t$):

$$B_t = B_s + (B_t - B_s) \,, \tag{1}$$

  where B_s is known at time s and $B_t - B_s$ is a random variable independent of the history of the process up until time s. In particular, $B_t - B_s$ is independent of B_s (this is a consequence of the independent increments property of Bm).
- Properties of the sBm B:
- (1) The sBm or the Bm are Gaussian processes
- (2) $cov\,(B_t, B_s) = \min\{t, s\}$
- (3) B is a Markov process
- (4) B is a martingale: we will clarify this property in class 2.

- (5) B returns infinitely often to 0 (or to any other level $a \in \mathbb{R}$).
- (6) (scaling property of Bm or self-similar property): If B_1 is a stochastic process defined by $B_1(t) := \frac{1}{\sqrt{c}} B_{ct}$, with $c > 0$, then B_1 is also a sBm.
- (7) (time inversion property): If If B_2 is a stochastic process defined by $B_2(t) := tB_{(1/t)}$ then B_2 is also a sBm.

Properties of Brownian motion

- A Gaussian process is essentially a process where its random variables are Gaussian random variables: this is clear for sBm by condition 4. of the definition.
- A Gaussian stochastic process is completely determined by its expectation and covariance function (as a normal r.v. is determined by its expectation and variance).
- If we know that a stochastic process has Gaussian increments and we know the first two moments of these increments, then we can determine all the statistical properties of the process.
- So, in order to prove that a stochastic process is a sBm, we only need to compute the expectation and the covariance function for the process and prove that they are equal to 0 and equal to the coveriance function given by property (2).

Properties of Brownian motion

- Proof of property (2): Let $s < t$. Then, using (1),

$$\begin{aligned} cov(B_t, B_s) &= cov[B_s + (B_t - B_s), B_s] \\ &= cov(B_s, B_s) + cov(B_t - B_s, B_s) \\ &= \mathbb{E}\left[B_s^2\right] + 0 = s. \end{aligned}$$

- Recall: X is a Markov process if the probability of obtaining a state at time t depends only of the state of the process at the previous last observed instant t_k and not from the previous history, i.e., if $t_1 < t_2 < \cdots < t_k < t$, then

$$\begin{aligned} & P[a < X_t < b | X_{t_1} = x_1, X_{t_2} = x_2, \ldots, X_{t_k} = x_k] \\ & = P[a < X_t < b | X_{t_k} = x_k]. \end{aligned}$$

Markov process with discrete state space: Markov chain. Markov process with continuous state space and continuous time: diffusion process.

- Proof of property (3)

$$\begin{aligned} & P[a < B_t < b | B_{t_1} = x_1, B_{t_2} = x_2, \ldots, B_{t_k} = x_k] \\ & = P[a - x_k < B_t - B_{t_k} < b - x_k | B_{t_k} = x_k], \end{aligned}$$

by the independent increments property of sBm (condition 2. of the definition).

- Proof of property (6): Clearly, (1) $B_1(0) = \frac{1}{\sqrt{c}} B_{ct} = \frac{1}{\sqrt{c}} B_0 = 0$, (2) $B_1(t) - B_1(s) = \frac{1}{\sqrt{c}}(B_{ct} - B_{cs})$ is independent of $\{B_{cu}, u \leq s\}$ by the independent increment property of the sBm. Therefore $B_1(t) - B_1(s)$ is independent of $\{B_1(u), u \leq s\}$ and B_1 has independent increments. (3 and 4) The distribution of $B_1(t) - B_1(s) = \frac{1}{\sqrt{c}}(B_{ct} - B_{cs}) \sim \frac{1}{\sqrt{c}} N(0, ct - cs) \sim N(0, t - s)$ and threfore B_1 has stationary increments.(5) $B_1(t) = \frac{1}{\sqrt{c}} B_{ct}$ has continuous sample paths because B_{ct} has continuous sample paths.
- Exercise: Prove the time inversion property (property 7) by computing the expectation and the covariance function of B_2.

Properties of Brownian motion

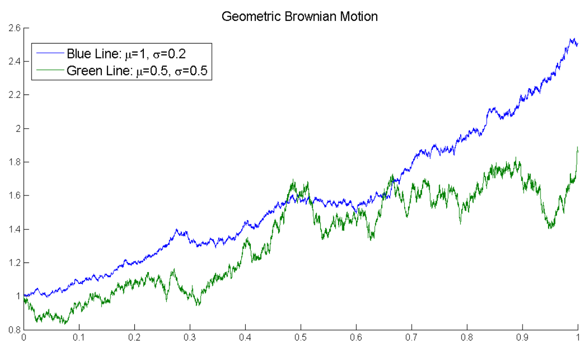
- (8) Property of non-differentiability of sample paths: The sample paths of a Bm are not differentiable anywhere a.s. (with probability 1).
- We will prove a weaker result:
- For any fixed time t_0, the probability that the sample path of a sBm is differentiable at t_0 is 0.
- **Proof:** Let us assume $t_0 = 0$ (the proof can be generalized to any t_0).If B has derivative a at 0 then:

$$a - \delta < \frac{B_t - B_0}{t} < a + \delta$$

for t small enough. This means that (with variable change: $s = \frac{1}{t}$) we have $a - \delta < sB_{(1/s)} < a + \delta$ and by the time inversion property (7), $sB_{(1/s)}$ is a sBm, so if we make $t \longrightarrow 0$ then $s \longrightarrow +\infty$ and the probability that a sBm remains confined between $a - \delta$ and $a + \delta$ when $s \longrightarrow +\infty$ is zero.

Geometrical Brownian motion

- The Bm is not very useful for modeling market prices (at the long run) since it can take negative values and the Bm model would imply that the sizes of price movements are independent of the level of the prices.
- A more useful and realistic model is the geometrical Brownian motion (gBm):
$$S_t = e^{W_t},$$
where W is a Bm $W_t = W_0 + \mu t + \sigma B_t$.
- S_t is lognormally distributed with mean $W_0 + \mu t$ and variance $\sigma^2 t$, i.e, the $\log(S_t) \sim N(W_0 + \mu t, \sigma^2 t)$.



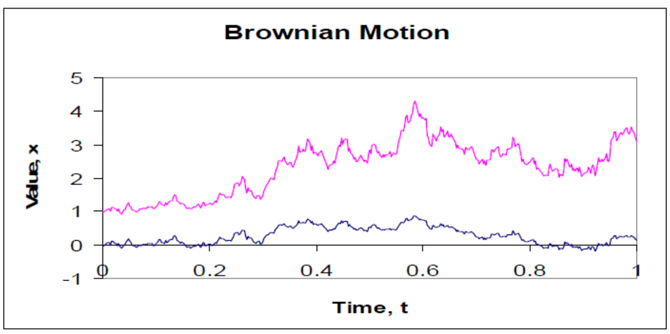


Figure 7.2: a Brownian motion $W_t=B_t+t$ and a geometric BM, $S_t=exp(W_t)$

Properties of geometrical Brownian motion

- $S_t \geq 0$ for all $t$and

$$\mathbb{E}\left[S_t\right] = \exp\left[(W_0 + \mu t) + \frac{1}{2}\sigma^2 t\right],$$
$$\mathrm{var}\left[S_t\right] = \left[\mathbb{E}\left[S_t\right]\right]^2 \left\{\exp\left[\sigma^2 t\right] - 1\right\}.$$

- In the Black-Scholes formula, the underlying asset price follows gBm.
- gBm properties are less helpful than those of Bm: increments of S are neither independent nor stationary.
- Analysis of gBm S: (1) take the logarithm of the observations; (2) Use techniques for the Bm.
- Log-return of a time series under gBm:

$$\log \frac{S_t}{S_s} = \log \frac{e^{W_t}}{e^{W_s}} = W_t - W_s.$$

Therefore, the log-returns (and the returns themselves) are independent over disjoint time periods.