Models in Finance - Class 2

Master in Actuarial Science

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Martingales

- Idea: a martingale is a stochastic process for which its "current value" is the "optimal estimator" of its expected "future value". Or:
- Given the stochastic process $\{M_j, j \in \mathbb{N}\}$ and the information $\mathcal{F}_n$ at instant n, then M_n is the best estimator for M_{n+1}.
- A martingale has "no drift" and its expected value remains constant in time.
- Martingale theory is fundamental in modern financial theory: the modern theory of pricing and hedging of financial derivatives is based on martingale theory.

Conditional expectation

- Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{B} \subset \mathcal{F}$ be a σ-algebra.

Definition

The conditional expectation of the integrable r.v. X given $\mathcal{B}$ (or $E(X|\mathcal{B})$) is an integral random variable Z such that

1. Z is $\mathcal{B}$-measurable
2. For each $A \in \mathcal{B}$, we have

$$E\left(Z\mathbf{1}_A\right) = E\left(X\mathbf{1}_A\right) \tag{1}$$

- If X is integrable (i.e., $E\left[|X|\right] < +\infty$) then $Z = E(X|\mathcal{B})$ exists and is unique (a.s.)

Conditional expectation

- Properties:

1.
$$E(aX + bY|\mathcal{B}) = aE(X|\mathcal{B}) + bE(Y|\mathcal{B}). \tag{2}$$

2.
$$E\left(E(X|\mathcal{B})\right) = E\left(X\right). \tag{3}$$

3. If X and the σ-algebra $\mathcal{B}$ are independent then:

$$E(X|\mathcal{B}) = E\left(X\right) \tag{4}$$

4. If X is $\mathcal{B}$-measurable (or if $\sigma(X) \subset \mathcal{B}$) then:

$$E(X|\mathcal{B}) = X. \tag{5}$$

5. If Y is $\mathcal{B}$-measurable (or if $\sigma(X) \subset \mathcal{B}$) then

$$E(YX|\mathcal{B}) = YE(X|\mathcal{B}) \tag{6}$$

6. Given two σ-algebras $\mathcal{C} \subset \mathcal{B}$ then

$$E(E(X|\mathcal{B})|\mathcal{C}) = E(E(X|\mathcal{C})|\mathcal{B}) = E(X|\mathcal{C}) \tag{7}$$

7. Consider r.v. X and Z such that Z is $\mathcal{B}$-measurable and X is independent of $\mathcal{B}$. Let $h(x,z)$ be a measurable functions such that $h(X,Z)$ is an integrable r.v.. Then

$$E(h(X,Z)|\mathcal{B}) = E(h(X,z))|_{z=Z}. \tag{8}$$

Note: First we compute $E(h(X,z))$ for a z fixed value of the r.v.. Z and then we replace z by Z.

Conditional expectation

- Given several r.v. $Y_1, Y_2, \ldots, Y_n$, we can consider the conditional expectation
$$E[X|Y_1, Y_2, \ldots, Y_n] = E[X|\beta],$$
where β is the σ-algebra generated by $Y_1, Y_2, \ldots, Y_n$.
- Note: The σ-algebra generated by a r.v. X is given by sets of the form $\sigma(X) := \left\{X^{-1}(B) : B \in \mathcal{B}_{\mathbb{R}}\right\}$.
- Important property: Let $\underline{Y}=(Y_1, Y_2, \ldots, Y_n)$ (notation). Then
$$E[E[X|\underline{Y}]] = E[X].$$
- Very important property (it is the reason why conditional expectation is so important): $E[X|\underline{Y}]$ is the optimal estimator of X based on $\underline{Y}$ in the sense that for every function h, we have:

$$E\left\{(X - E[X|\underline{Y}])^2\right\} \leq E\left\{(X - h(\underline{Y}))^2\right\}. \tag{9}$$

Martingales

- Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{\mathcal{F}_n, n \geq 0\}$ be a sequence of σ-algebras such that
$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_n \subset \cdots \subset \mathcal{F} \tag{10}$$
The sequence $\{\mathcal{F}_n, n \geq 0\}$ is called a filtration
- Filtration $\approx$ information flow.

Definition

$M = \{M_n; n \geq 0\}$ (in discrete time) is a martingale with respect to filtration $\{\mathcal{F}_n, n \geq 0\}$ if:

1. For each n, M_n is a $\mathcal{F}_n$-measurable r.v. (i.e., M is a stochastic process adapted to the filtration $\{\mathcal{F}_n, n \geq 0\}$).
2. For each n, $E[|M_n|] < \infty$.
3. For each n, we have:
$$E[M_{n+1}|\mathcal{F}_n] = M_n. \tag{11}$$

Martingales

- If we consider the filtration $\mathcal{F}_n = \sigma(M_0, M_1, \ldots, M_n)$, then we say that $M = \{M_n; n \geq 0\}$ is a martingale (with respect to this filtration) if

1. For each n, $E[|M_n|] < \infty$.
2. For each n, we have:
$$E[M_{n+1}|\mathcal{F}_n] = M_n. \tag{12}$$

- Properties: It is easy to show that if $M = \{M_n; n \geq 0\}$ is a martingale then

1. $E[M_n] = E[M_0]$ for all $n \geq 1$.
2. $E[M_n|\mathcal{F}_k] = M_k$ for all $n \geq k$.

- Exercise: Prove properties 1. and 2. above.

Martingales

- idea: the "current value" M_k of a martingale is the "optimal estimator" of its "future value" M_n.
- martingale and risk neutral probability measure: If the discounted price of a financial asset is a martingale when calculated using a particular probability distribution, then this probability distribution is called a "risk-neutral" probability measure (meaning that the price has no "drift").

- Example: Assume that share S has a price process S_t and a discounted price process

$$\widetilde{S}_t = e^{-rt} S_t, \tag{13}$$

where r is the risk-free interest rate. If we assume that for a probability measure Q, the process $\widetilde{S}_t$ is a martingale, then under Q, we have that

$$E_Q\left[\widetilde{S}_{n+1} | \widetilde{S}_0, \widetilde{S}_1, \ldots, \widetilde{S}_n\right] = \widetilde{S}_n.$$

Since $\widetilde{S}_n$ is known (it is measurable) with respect to $\sigma\left(\widetilde{S}_0, \widetilde{S}_1, \ldots, \widetilde{S}_n\right)$, then by property (5), we have:

$$E_Q\left[\frac{e^{-r(n+1)} S_{n+1}}{e^{-rn} S_n} | \widetilde{S}_0, \widetilde{S}_1, \ldots, \widetilde{S}_n\right] = 1$$

$$\Longleftrightarrow E_Q\left[\frac{S_{n+1}}{S_n} | S_0, S_1, \ldots, S_n\right] = e^r.$$

Therefore, the expected return in period from time n to time $n+1$ is the risk-free rate: that is why the distribution Q is called risk-neutral

Martingales in continuous time

- Probability space $(\Omega, \mathcal{F}, P)$ and family of σ-algebras $\{\mathcal{F}_t, t \geq 0\}$ such that
$$\mathcal{F}_s \subset \mathcal{F}_t, \quad 0 \leq s \leq t. \tag{14}$$
The family $\{\mathcal{F}_t, t \geq 0\}$ is called a filtration
- Let $\mathcal{F}_t^X$ be the σ-algebra generated by process X on the interval $[0, t]$, i.e. $\mathcal{F}_t^X = \sigma(X_s, 0 \leq s \leq t)$. Then $\mathcal{F}_t^X$ is the "information generated by X on interval $[0, t]$" or "history of the process X up until time t".
- $A \in \mathcal{F}_t^X$ means that is possible to decide if event A occurred or not, based on the observation of the paths of the process X on $[0, t]$.
- Example: If $A = \{\omega : X(5) > 1\}$ then $A \in \mathcal{F}_5^X$ but $A \notin \mathcal{F}_4^X$.
- A stochastic process Y is said to be adapted to the filtration $\{\mathcal{F}_t, t \geq 0\}$ if Y_t is $\mathcal{F}_t$ measurable for all t.
- If $\mathcal{F}_t^X = \sigma(X_s, 0 \leq s \leq t)$ is the filtration generated by X, then any function of X_t is adapted to $\mathcal{F}_t^X$.

Martingales in continuous time

- Key properties:
  1. $E[X|\mathcal{F}_t]$ is the optimal estimator of X among all $\mathcal{F}_t$-measurable random variables with finite expectation, or equivalently
  $$E\{(X - E[X|\mathcal{F}_t])Y\} = 0 \tag{15}$$
  for all $\mathcal{F}_t$-measurable bounded random variables Y.
  2. $E\{E[X|\mathcal{F}_t]\} = E[X]$.
  3. If X is $\mathcal{F}_t$-measurable then $E[X|\mathcal{F}_t] = X$.
  4. If Y is $\mathcal{F}_t$-measurable and bounded then $E[XY|\mathcal{F}_t] = YE[X|\mathcal{F}_t]$.
  5. If X is independent of $\mathcal{F}_t$ then $E[X|\mathcal{F}_t] = E[X]$.

Martingales in continuous time

Definition

A stochastic process $M = \{M_t; t \geq 0\}$ is a martingale with respect to the filtration $\{\mathcal{F}_t, t \geq 0\}$ if:

1. For each $t \geq 0$, M_t is a $\mathcal{F}_t$-measurable r.v. (i.e., M is adapted to $\{\mathcal{F}_t, t \geq 0\}$).
2. For each $t \geq 0$, $E[|M_t|] < \infty$.
3. For each $s \leq t$,
$$E[M_t|\mathcal{F}_s] = M_s. \tag{16}$$

Martingales in continuous time

- cond. (3) $\Longleftrightarrow E[M_t - M_s|\mathcal{F}_s] = 0$.
- If $t \in [0, T]$ then $M_t = E[M_T|\mathcal{F}_t]$.
- cond. (3) $\Longrightarrow E[M_t] = E[M_0]$ for all t.

Martingales in continuous time

- Consider a Bm $B = \{B_t; t \geq 0\}$ defined on $(\Omega, \mathcal{F}, P)$and

$$\mathcal{F}_t^B = \sigma \{B_s, s \leq t\}. \quad (17)$$

Proposition: The following processes are $\mathcal{F}_t^B$-martingales:

1. B_t.
2. $B_t^2 - t$.
3. $\exp\left(aB_t - \frac{a^2 t}{2}\right)$.(Exercise: prove that this process is a martingale).

Martingales in continuous time

Proof.

1. B_t is $\mathcal{F}_t^B$-measurable and therefore it is adapted. $E[|B_t|] < \infty$ (why?)Moreover $B_t - B_s$ is independent of $\mathcal{F}_s^B$ (why?).Hence (why?)

$$E\left[B_t - B_s | \mathcal{F}_s^B\right] = E[B_t - B_s] = 0.$$

2. Clearly, $B_t^2 - t$ is $\mathcal{F}_t^B$-measurable and adapted (why?) and $E\left[\left|B_t^2 - t\right|\right] < \infty$. By the properties of the conditional expectation

$$\begin{aligned} E\left[B_t^2 - t | \mathcal{F}_s^B\right] &= E\left[(B_t - B_s + B_s)^2 | \mathcal{F}_s^B\right] - t \\ &= E\left[(B_t - B_s)^2\right] + 2B_s E\left[B_t - B_s | \mathcal{F}_s^B\right] + B_s^2 - t \\ &= t - s + B_s^2 - t = B_s^2 - s. \end{aligned}$$

□

Martingales in continuous time

- Exercise: Prove that $\exp\left(aB_t - \frac{a^2 t}{2}\right)$ is a $\left\{\mathcal{F}_t^B, t \geq 0\right\}$-martingale.