

Models in Finance - Class 3

Master in Actuarial Science

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Stochastic integrals

- Motivation : Consider a "differential equation" with "noise" of type:

$$\frac{dX}{dt} = b(t, X_t) + \sigma(t, X_t) \frac{dB_t}{dt}.$$

- " $\frac{dB_t}{dt}$ " is stochastic "noise". Does not exist in classical sense since B is not differentiable.
- "Stochastic differential equation" (SDE) in integral form :

$$X_t = X_0 + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dB_s \quad (1)$$

- How to define the integral:

$$\int_0^T u_s dB_s \quad ? \quad (2)$$

where B is a Brownian motion and u is an appropriate adapted process.

- Note: the SDE's that we deal with are the continuous time versions of the equations used to define time series (processes in discrete time). Example: a zero-mean random walk can be defined by:

$$X_t = X_{t-1} + \sigma Z_t,$$

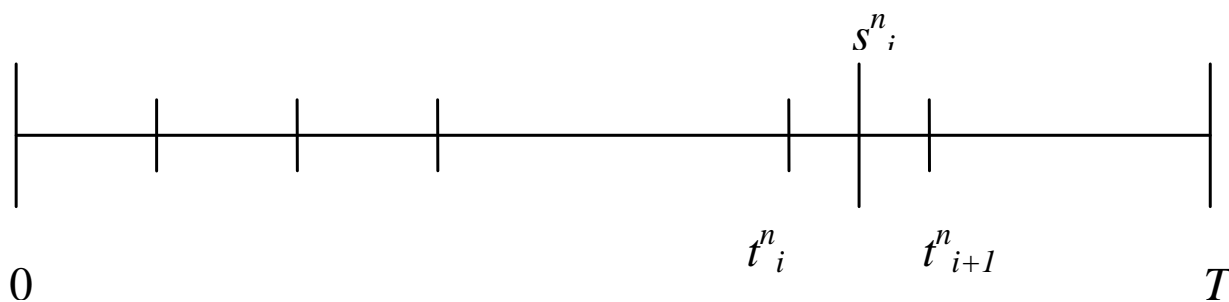
where Z_t is a standard normal r.v. (the Z_i variables are called white noise). This equation is a stochastic difference equation and is equivalent to $\Delta X_t = \sigma Z_t$. Its solution is $X_t = X_0 + \sigma \sum_{s=1}^t Z_s$.

- In continuous time, the analog of a zero-mean random walk is a zero-mean Brownian motion W_t .

- First strategy: Consider the integral (2)
- Consider a sequence of partitions of $[0, T]$ and a sequence of points:

$$\begin{aligned} \tau_n: 0 &= t_0^n < t_1^n < t_2^n < \dots < t_{k(n)}^n = T \\ s_n: t_i^n &\leq s_i^n \leq t_{i+1}^n, \quad i = 0, \dots, k(n) - 1, \end{aligned}$$

such that $\limsup_{n \rightarrow \infty} \max_i (t_{i+1}^n - t_i^n) = 0$.



Riemann-Stieltjes (R-S) integral:

$$\int_0^T f dg := \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} f(s_i^n) \Delta g_i,$$

where $\Delta g_i := g(t_{i+1}^n) - g(t_i^n)$, if the limit exists and is independent of the sequences τ_n and s_n .

- If g is a differentiable function and f is continuous the (R-S) integral is well defined: $\int_0^T f(t) dg(t) = \int_0^T f(t) g'(t) dt$.
- The (R-S) $\int_0^T f dg$ exists if f is continuous and g bounded total variation, i.e.,

$$\sup_{\tau_n} \sum_j |\Delta g_i| < \infty.$$

- In the Bm case B , it is clear that $B'(t)$ does not exist, so we cannot define the path integral:

$$\int_0^T u_t(\omega) dB_t(\omega) \neq \int_0^T u_t(\omega) B'_t(\omega) dt$$

- We can prove that Bm total variation is not bounded and therefore we cannot define the integral (R-S) $\int_0^T u_t(\omega) dB_t(\omega)$.
- If u has sample paths of class C^1 , then integrating by parts, the (R-S) integral exists and we can compute:

$$\int_0^T u_t(\omega) dB_t(\omega) = u_T(\omega) B_T(\omega) - \int_0^T u'_t(\omega) B_t(\omega) dt.$$

- Problem: The integral $\int_0^T B_t(\omega) dB_t(\omega)$ does not exist as a R-S integral. We need to consider processes u with sample paths more irregular than C^1 trajectories. How to define the integral (2) for these processes?

- We will construct the stochastic integral $\int_0^T u_t dB_t$ using a probabilistic approach.

Definition

Consider processes u of class $L^2_{a,T}$, which is defined as the class of processes $u = \{u_t, t \in [0, T]\}$, such that:

- ① u is adapted and measurable.
- ② $E \left[\int_0^T u_t^2 dt \right] < \infty$.

- Condition 1. allows us to show that $\int_0^t u_s ds$ is \mathcal{F}_t -measurable
- Condition 2. allows us to show that u as a map of two variables t and ω belongs to the space $L^2([0, T] \times \Omega)$ and that:

$$E \left[\int_0^T u_t^2 dt \right] = \int_0^T E [u_t^2] dt = \int_{[0, T] \times \Omega} u_t^2(\omega) dt P(d\omega).$$

- idea: we will define $\int_0^T u_t dB_t$ for $u \in L^2_{a, T}$ as a limit in mean-square (i.e., a limit in $L^2(\Omega)$) of integrals of simple processes.

Stochastic Itô integral for simple processes

Definition

$u \in \mathcal{S}$ (set of simple processes in $[0, T]$) is called a simple process if

$$u_t = \sum_{j=1}^n \phi_j \mathbf{1}_{(t_{j-1}, t_j]}(t), \quad (3)$$

where $0 = t_0 < t_1 < \dots < t_n = T$, and the r.v. ϕ_j are square-integrables ($E[\phi_j^2] < \infty$) and $\mathcal{F}_{t_{j-1}}$ -measurable

Definition

If u is a simple process of form (3) ($u \in \mathcal{S}$) then the stochastic Itô integral of u with respect to Bm B is:

$$\int_0^T u_t dB_t := \sum_{j=1}^n \phi_j (B_{t_j} - B_{t_{j-1}}).$$

Example

Consider the simple process

$$u_t = \sum_{j=1}^n B_{t_{j-1}} \mathbf{1}_{(t_{j-1}, t_j]}(t).$$

Then

$$\int_0^T u_t dB_t = \sum_{j=1}^n B_{t_{j-1}} (B_{t_j} - B_{t_{j-1}}).$$

Then (why?)

$$\begin{aligned} E \left[\int_0^T u_t dB_t \right] &= \sum_{j=1}^n E \left[B_{t_{j-1}} (B_{t_j} - B_{t_{j-1}}) \right] \\ &= \sum_{j=1}^n E \left[B_{t_{j-1}} \right] E \left[B_{t_j} - B_{t_{j-1}} \right] = 0. \end{aligned}$$

Proposition: (Isometry property or norm preservation property). Let $u \in \mathcal{S}$. Then:

$$E \left[\left(\int_0^T u_t dB_t \right)^2 \right] = E \left[\int_0^T u_t^2 dt \right] = \int_0^T E \left[u_t^2 \right] dt. \quad (4)$$

Proof.

With $\Delta B_j := B_{t_j} - B_{t_{j-1}}$, we have (Exercise (homework): justify all the steps in this proof):

$$\begin{aligned} E \left[\left(\int_0^T u_t dB_t \right)^2 \right] &= E \left[\left(\sum_{j=1}^n \phi_j \Delta B_j \right)^2 \right] \\ &= \sum_{j=1}^n E \left[\phi_j^2 (\Delta B_j)^2 \right] + 2 \sum_{i < j} E \left[\phi_i \phi_j \Delta B_i \Delta B_j \right]. \end{aligned}$$

□

Proof.

(cont.) Note that since $\phi_i\phi_j\Delta B_i$ is \mathcal{F}_{j-1} -measurable and ΔB_j is independent of \mathcal{F}_{j-1} , then

$$\sum_{i < j}^n E [\phi_i\phi_j\Delta B_i\Delta B_j] = \sum_{i < j}^n E [\phi_i\phi_j\Delta B_i] E [\Delta B_j] = 0.$$

On the other hand, since ϕ_j^2 is \mathcal{F}_{j-1} -measurable and ΔB_j is independent of \mathcal{F}_{j-1} ,

$$\begin{aligned} \sum_{j=1}^n E [\phi_j^2 (\Delta B_j)^2] &= \sum_{j=1}^n E [\phi_j^2] E [(\Delta B_j)^2] \\ &= \sum_{j=1}^n E [\phi_j^2] (t_j - t_{j-1}) = \\ &= E \left[\int_0^T u_t^2 dt \right]. \end{aligned}$$

- Other properties of $\int_0^T u_t dB_t$ for $u \in \mathcal{S}$:

- ① Linearity: If $u, v \in \mathcal{S}$:

$$\int_0^T (au_t + bv_t) dB_t = a \int_0^T u_t dB_t + b \int_0^T v_t dB_t. \quad (5)$$

- ② Zero mean:

$$E \left[\int_0^T u_t dB_t \right] = 0. \quad (6)$$

Exercise: Prove the property 2.

Exercise: Compute $\int_0^5 f(s) dB_s$ with $f(s) = 1$ if $0 \leq s \leq 2$ and $f(s) = 4$ if $2 < s \leq 5$ and what is the distribution of the resulting r.v.?

Itô integral

Lemma

If $u \in L^2_{a,T}$ then exists a sequence of simple processes $\{u^{(n)}\}$ such that

$$\lim_{n \rightarrow \infty} E \left[\int_0^T |u_t - u_t^{(n)}|^2 dt \right] = 0. \quad (7)$$

Proof: see Oksendal's book or Nualart lecture notes:

<http://www.math.ku.edu/~nualart/StochasticCalculus.pdf>

Definition

The Itô stochastic integral of $u \in L^2_{a,T}$ is defined as the limit (in the $L^2(\Omega)$ sense):

$$\int_0^T u_t dB_t = \lim_{n \rightarrow \infty} (L^2) \int_0^T u_t^{(n)} dB_t, \quad (8)$$

where $\{u^{(n)}\}$ is a sequence of simple processes satisfying (7).

Properties of the Itô integral

- Properties of the Itô integral $\int_0^T u_t dB_t$ for $u \in L^2_{a,T}$.

- 1 Isometry (or norm preservation):

$$E \left[\left(\int_0^T u_t dB_t \right)^2 \right] = E \left[\int_0^T u_t^2 dt \right] = \int_0^T E \left[u_t^2 \right] dt. \quad (9)$$

- 2 Zero mean:

$$E \left[\int_0^T u_t dB_t \right] = 0 \quad (10)$$

- 3 Linearity:

$$\int_0^T (au_t + bv_t) dB_t = a \int_0^T u_t dB_t + b \int_0^T v_t dB_t. \quad (11)$$

- 4 The process $\left\{ \int_0^t u_s dB_s, t \geq 0 \right\}$ is a martingale.

- 5 The sample paths of $\left\{ \int_0^t u_s dB_s, t \geq 0 \right\}$ are continuous.

Example

Let us show that

$$\int_0^T B_t dB_t = \frac{1}{2} B_T^2 - \frac{1}{2} T.$$

Since $u_t = B_t$, let us consider the sequence of simple processes

$$u_t^n = \sum_{j=1}^n B_{t_{j-1}^n} \mathbf{1}_{(t_{j-1}^n, t_j^n]}(t),$$

with $t_j^n := \frac{j}{n} T$.

Example

(cont.)

$$\begin{aligned}
 \int_0^T B_t dB_t &= \lim_{n \rightarrow \infty} (L^2) \int_0^T u_t^{(n)} dB_t = \\
 &= \lim_{n \rightarrow \infty} (L^2) \sum_{j=1}^n B_{t_{j-1}^n} (B_{t_j^n} - B_{t_{j-1}^n}) \\
 &= \lim_{n \rightarrow \infty} (L^2) \frac{1}{2} \sum_{j=1}^n \left[(B_{t_j^n}^2 - B_{t_{j-1}^n}^2) - (B_{t_j^n} - B_{t_{j-1}^n})^2 \right] \\
 &= \frac{1}{2} (B_T^2 - T),
 \end{aligned}$$

where we used: $E \left[\left(\sum_{j=1}^n (\Delta B_{t_j^n})^2 - T \right)^2 \right] = 0$ and

$$\frac{1}{2} \sum_{j=1}^n (B_{t_j^n}^2 - B_{t_{j-1}^n}^2) = \frac{1}{2} B_T^2.$$

- Let us prove that $E \left[\left(\sum_{j=1}^n (\Delta B_{t_j^n})^2 - T \right)^2 \right] = 0$. Using the independence of increments and $E \left[(\Delta B_{t_j^n})^2 \right] = \Delta t_j^n$, then

$$\begin{aligned}
 E \left[\left(\sum_{j=1}^n (\Delta B_{t_j^n})^2 - T \right)^2 \right] &= E \left[\left(\sum_{j=1}^n \left[(\Delta B_{t_j^n})^2 - \Delta t_j^n \right] \right)^2 \right] \\
 &= \sum_{j=1}^n E \left[(\Delta B_{t_j^n})^2 - \Delta t_j^n \right]^2.
 \end{aligned}$$

Using the fact that $E \left[(B_t - B_s)^{2k} \right] = \frac{(2k)!}{2^k \cdot k!} (t - s)^k$, then

$$\begin{aligned}
 E \left[\left(\sum_{j=1}^n (\Delta B_{t_j^n})^2 - T \right)^2 \right] &= \sum_{j=1}^n \left[3\Delta t_j^n - 2(\Delta t_j^n)^2 + (\Delta t_j^n)^2 \right] \\
 &= 2 \sum_{j=1}^n (\Delta t_j^n)^2 = 2T \sup_j |\Delta t_j^n| \xrightarrow{n \rightarrow \infty} 0.
 \end{aligned}$$

- Note: By formula $E \left[(B_t - B_s)^{2k} \right] = \frac{(2k)!}{2^k \cdot k!} (t - s)^k$ we have that

$$\begin{aligned} \text{Var} \left[(\Delta B)^2 \right] &= E \left[(\Delta B)^4 \right] - \left(E \left[(\Delta B)^2 \right] \right)^2 \\ &= 3 (\Delta t)^2 - (\Delta t)^2 = 2 (\Delta t)^2. \end{aligned}$$

We also know that

$$E \left[(\Delta B)^2 \right] = \Delta t.$$

Therefore, if Δt is small, the variance of $(\Delta B)^2$ is very small when compared with its expected value \implies therefore when $\Delta t \rightarrow 0$ or " $\Delta t = dt$ ", we have:

$$(dB_t)^2 \approx dt. \tag{12}$$