Lévy processes and applications - A general introduction

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Outline

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Programme

- General introduction and Imperfections of the Black-Scholes model.
- Lévy processes. Definitions, examples and Basic properties
- Stochastic calculus for Lévy processes.
- Lévy processes in finance
- Option pricing with Lévy models

Bibliography

Main:

- Cont, R. and Tankov, P. (2003), Financial modelling with Jump Processes, Chapman & Hall / CRC Press
- Applebaum, D. (2004), Lévy Processes and Stochastic Calculus, Cambridge University Press

Other:

- Oksendal, B. and Sulem, A. (2007), Applied Stochastic Control of Jump Diffusions, 2nd. Edition, Springer.
- Papapantoleon, A. (2008), An introduction to Lévy processes with applications in finance. Lecture notes, TU Vienna, 2008, http://arxiv.org/abs/0804.0482
- Sato, K.-I. (1999), Lévy Processes and Infinitely Divisible Distributions, Cambridge University Press
- Schoutens, W (2003), Lévy Processes in Finance, John Wiley & Sons

Assessment

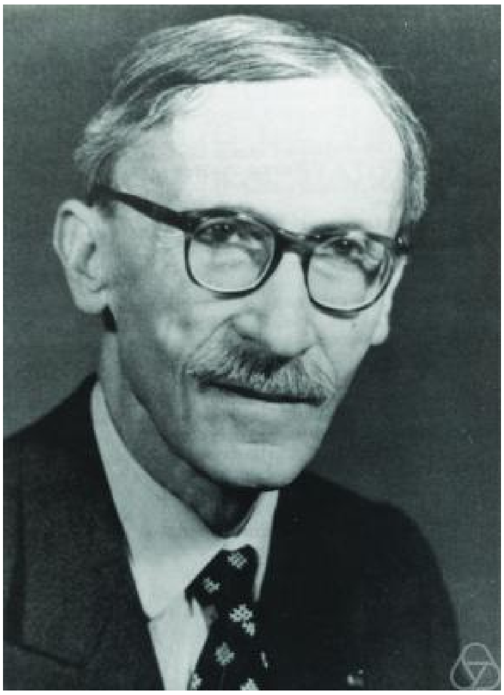
- The final grade, on a 0-20 scale, is awarded on the basis of a final written exam (50%) and a group assignment distributed during the semester (50%)
- Group assignment: the presentation and discussion of an important scientific paper in the field of Lévy processes and applications in finance
- Groups of 2 students

Introduction

- **Lévy process**: stochastic process with stationary and independent increments.
- The basic theory was developed by Paul Lévy (1886-1971) on the 1930s.
- Why the interest in Lévy Processes?
- Many interesting examples: Brownian motion, Poisson processes, jump-diffusion processes, subordinated processes, financial models, etc...
- Lévy processes are the simplest generic class of processes with continuous paths interspersed with random jumps at random times.
- Lévy processes are a natural subclass of semimartingales.
- A large class of Markov processes can be built as solutions of stochastic differential equations driven by Lévy noise.

- Lévy processes have a "robust structure": most applications deal with Lévy processes taking values in Euclidean space but this can be replaced by a Hilbert space or a Banach space (for SPDE’s).
- Applications:
  - Turbulence
  - Finance
  - Physics
- Main areas in Finance:
  - Option pricing in incomplete markets
  - Interest rate modelling
- Why in Finance?
  - Describe the observed reality in a more accurate way than the usual Brownian motion models: asset prices have jumps; Empirical distribution of the returns exhibits "fat tails" and skewness; the implied volatilities are constant neither across strike nor across maturities.

- Paul Lévy (1886-1971)



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- Aleksandr Khintchine (1894-1959)

Imperfections of the Black-Scholes model

- Asset price processes have jumps.
- Empirical distribution of asset returns exhibits fat tails and skewness.
- Implied volatilities are constant neither cross strike nor across maturities.

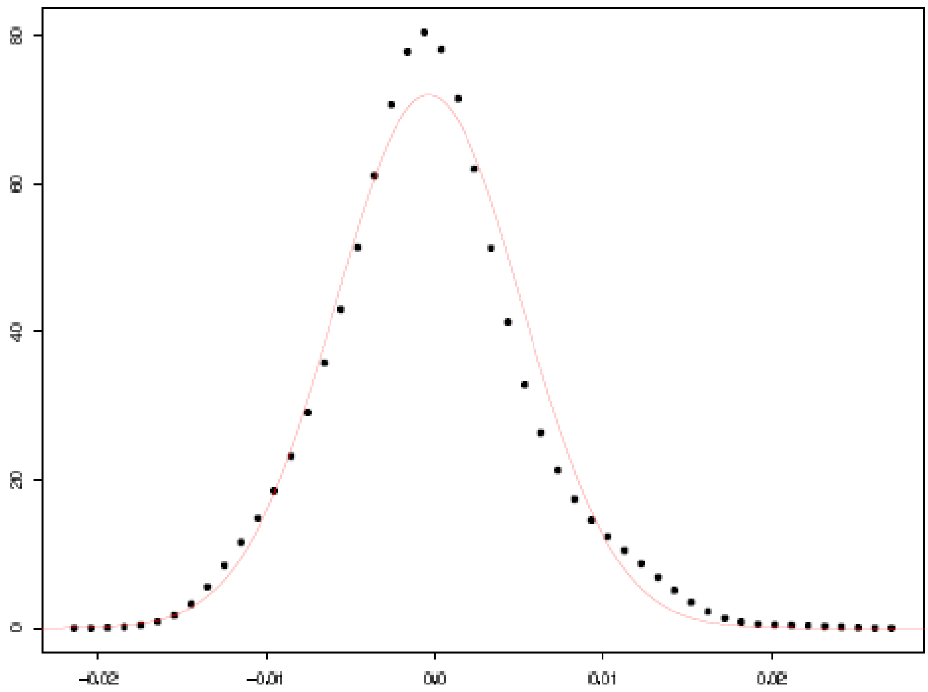


Figure: Empirical Distribution of daily log-returns for the GBP/USD exchange rate and fitted normal distribution

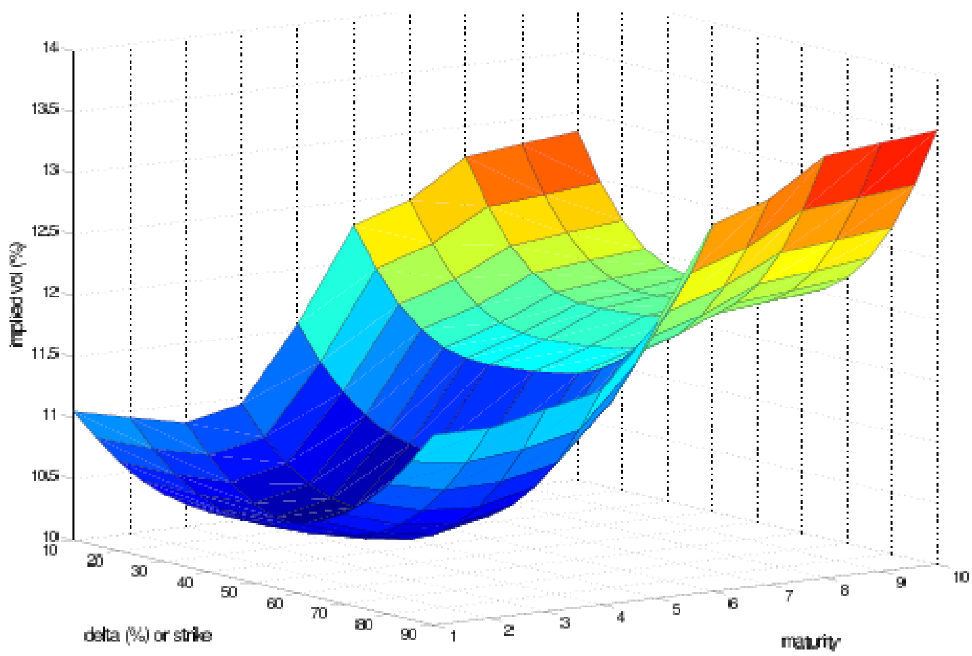


Figure: Implied volatilities of vanilla options on the EUR/USD exchange rate on November 5, 2001.

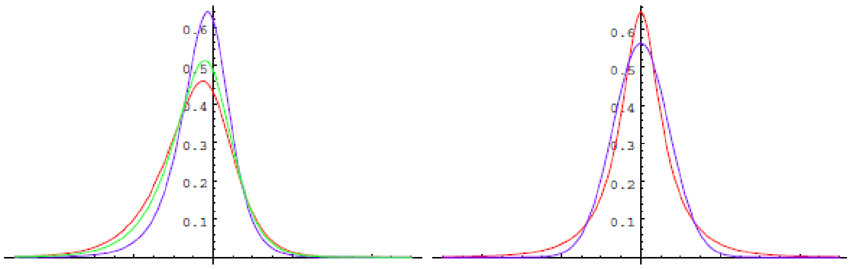


FIGURE 18.11. Densities of hyperbolic (red), NIG (blue) and hyperboloid distributions (left). Comparison of the GH (red) and Normal distributions (with equal mean and variance).

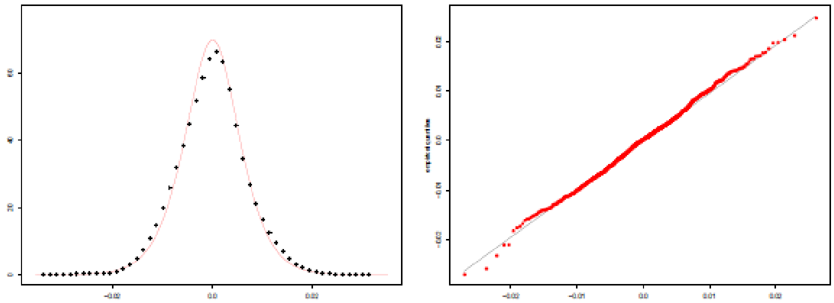


FIGURE 18.12. Empirical distribution and Q-Q plot of EUR/USD daily log-returns with fitted GH (red).

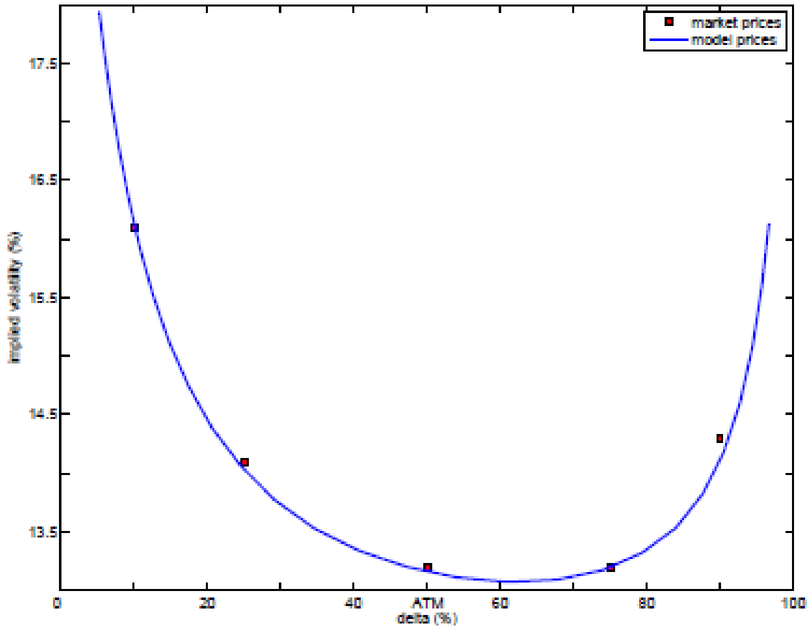


FIGURE 18.13. Implied volatilities of EUR/USD options and calibrated NIG smile.

Lévy Processes - Definition

Definition

A càdlág, adapted, stochastic process $L = \{L_t, t \in [0, T]\}$ is a **Lévy process** if
$L_0 = 0$ a.s. and
L has independent increments
L has stationary increments
L is stochastically continuous, i.e., for every $t \in [0, T]$ and $\varepsilon > 0$, we have

$$\lim_{s \to t} \mathbb{P}\left[|L_t - L_s| > \varepsilon\right] = 0.$$

- An example (jump-diffusion)

$$L_t = bt + \sigma W_t + \sum_{k=1}^{N_t} J_k - t\lambda m, \qquad (1)$$

where N is a Poisson process with parameter λ and $J = (J_k)_{k \geq 1}$ is a i.i.d. sequence with probab. distribution F and $\mathbb{E}[J] = m$.

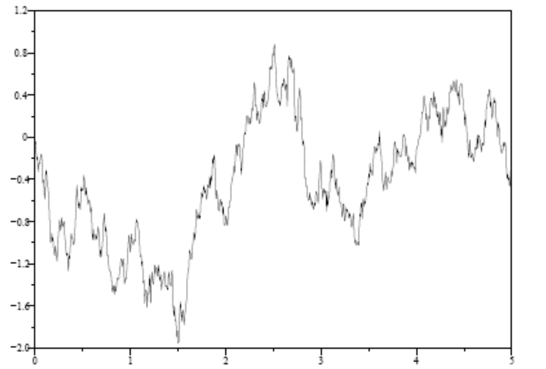
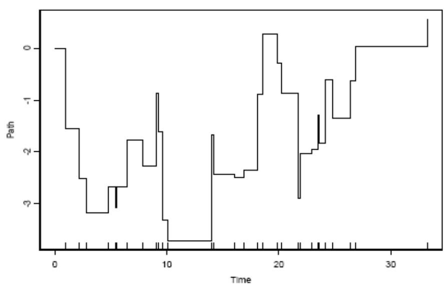


Figure 1 Simulation of standard Brownian motion

Figure:



summands($\lambda = 1$).

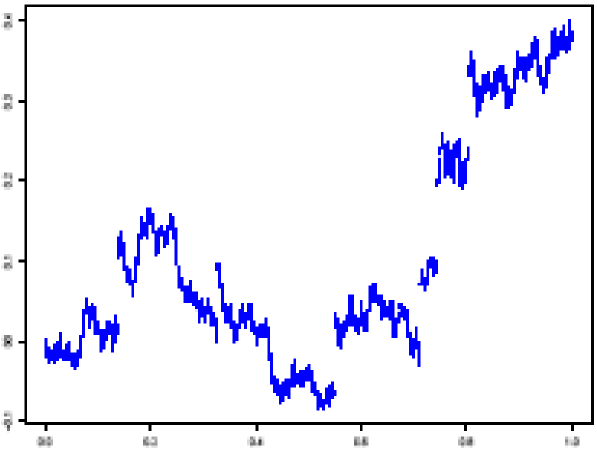


Figure: A jump-diffusion trajectory

Infinitely divisible distributions

- The characteristic function of the jump diffusion (1) is

$$\mathbb{E}\left[e^{iuL_t}\right] = \exp\left[t\left(iub - \frac{u^2\sigma^2}{2} + \int_{\mathbb{R}} \left(e^{iux} - 1 - iux\right)\lambda F(dx)\right)\right]. \qquad (2)$$

- Sketch of the proof:

$$\mathbb{E}\left[e^{iuL_t}\right] = \exp\left[iubt\right]\mathbb{E}\left[\exp\left[iu\sigma W_t\right]\right]\mathbb{E}\left[\exp\left[iu\sum_{k=1}^{N_t} J_k - iut\lambda m\right]\right].$$

$$\mathbb{E}\left[\exp\left[iu\sigma W_t\right]\right] = \exp\left[-\frac{1}{2}\sigma^2 u^2 t\right], \qquad W_t \sim N(0,t),$$

$$\mathbb{E}\left[\exp\left[iu\sum_{k=1}^{N_t} J_k\right]\right] = \exp\left[\lambda t \mathbb{E}\left[e^{iu\,J} - 1\right]\right], \qquad N_t \sim Po(\lambda t).$$

$$\mathbb{E}\left[e^{iuL_t}\right] = \exp\left[iubt - \frac{\sigma^2 u^2 t}{2}\right]\exp\left[\lambda t \int_{\mathbb{R}} \left(e^{iux} - 1 - iux\right)\lambda F(dx)\right].$$

Infinitely divisible distributions

Definition

The law P_X of a r.v. X is infinitely divisible if for all $n \in \mathbb{N}$, there exist i.i.d. random variables $X_1^{(1/n)}, X_2^{(1/n)}, \ldots, X_n^{(1/n)}$, such that:

$$X \overset{d}{=} X_1^{(1/n)} + X_2^{(1/n)} + \ldots + X_n^{(1/n)}.$$

- P_X is infinitely divisible if, for all $n \in \mathbb{N}$, exists a r.v. $X^{(1/n)}$ such that

$$\varphi_X(u) = \left(\varphi_{X^{(1/n)}}(u)\right)^n.$$

Example

(The Poisson Distribution): $X \sim Po(\lambda)$; $X^{(1/n)} \sim Po\left(\frac{\lambda}{n}\right)$.

$$\begin{aligned} \varphi_X(u) &= \exp\left(\lambda\left(e^{iu}-1\right)\right) \\ &= \left(\exp\left[\frac{\lambda}{n}\left(e^{iu}-1\right)\right]\right)^n = \left(\varphi_{X^{(1/n)}}(u)\right)^n. \end{aligned}$$

Lévy-Kintchine formula

Theorem

(Lévy Khintchine formula): P_X is infinitely divisible if and only if exists a triplet (b, c, ν), $b \in \mathbb{R}$, $c \geq 0$, where ν is a measure, $\nu(\{0\}) = 0$, $\int_{\mathbb{R}} \left(1 \wedge x^2\right) \nu(dx) < \infty$ and

$$\mathbb{E}\left[e^{iuX}\right] = \exp\left[ibu - \frac{u^2 c}{2} + \int_{\mathbb{R}} \left(e^{iux} - 1 - iux\mathbf{1}_{\{|x|<1\}}\right) \nu(dx)\right].$$

Characteristic triplet of a Lévy process

- The triplet (b, c, ν) is called the Lévy or characteristic triplet and the exponent

$$\psi(u) = ibu - \frac{u^2 c}{2} + \int_{\mathbb{R}} \left(e^{iux} - 1 - iux\mathbf{1}_{\{|x|<1\}}\right) \nu(dx)$$

  is called the Lévy or characteristic exponent.
- b is the drift term, c is the Gaussian or diffusion coefficient and ν is the Lévy measure.
- The r.v. L_t of the jump diffusion process (1) has infinitely divis. dist. and $b = bt$, $c = \sigma^2 t$ and $\nu = (\lambda F)\, t$.
- Consider a general Lévy process $L = \{L_t, t \in [0, T]\}$. Then

$$L_t = L_{\frac{t}{n}} + \left(L_{\frac{2t}{n}} - L_{\frac{t}{n}}\right) + \cdots + \left(L_t - L_{\frac{(n-1)t}{n}}\right).$$

  By the stationarity and independence of increments, $\left(L_{\frac{kt}{n}} - L_{\frac{(k-1)t}{n}}\right)$ is an iid sequence. Therefore, L_t has an infinitely divisible dist.

Lévy-Kintchine formula for a Lévy process

- The characteristic function of a Lévy process is given by the **Lévy-Khintchine formula** (infinitely divisible distribution):

$$\begin{aligned}\phi_u(t) &= \mathbb{E}\left[e^{iuL_t}\right] = \exp\{t\psi(u)\} \\ &= \exp\left\{t\left(ibu - \frac{u^2 c}{2} + \int_{-\infty}^{+\infty} \left(e^{iux} - 1 - iux\mathbf{1}_{\{|x|<1\}}\right) \nu(dx)\right)\right\},\end{aligned}$$

  where ν is the Lévy measure, (b, c, ν) is the triplet of characteristics of the Lévy process and $\psi(u)$ is the characteristic exponent of L_1.
- Every Lévy process can be associated with a infinitely divisible distribution.
- The opposite (Lévy-Itô decomposition) is also true. Given a r.v. X with infinitely divisible distrib., we can construct a Lévy process $L = \{L_t, t \in [0, T]\}$ such that the law of L_1 is the law of X.

Jumps of a Lévy process

- Jump process: $\Delta L = \{\Delta L_t, t \in [0, T]\}$, where

$$\Delta L_t = L_t - L_{t-}.$$

- By the stochastic continuity of L, for a fixed t, $\Delta L_t = 0$ a.s.
- It is possible that

$$\sum_{s \le t} |\Delta L_s| = \infty \quad \text{a.s.}$$

- However,

$$\sum_{s \le t} |\Delta L_s|^2 < \infty \quad \text{a.s.}$$

Poisson random measures

- Let $A \in \mathcal{B}(\mathbb{R} \setminus \{0\})$ such that $0 \in \overline{A}$. The Poisson random measure of the jumps:

$$\mu^L(\omega, t, A) = \#\{0 \le s \le t; \Delta L_s \in A\} = \sum_{s \le t} \mathbf{1}_A(\Delta L_s(\omega)).$$

- $\mu^L(\cdot, A)$ has independent and stationary increments.
- Hence, $\mu^L(\cdot, A)$ is a Poisson process and μ^L is called a Poisson random measure.
- The measure ν defined on $\mathcal{B}(\mathbb{R} \setminus \{0\})$ by

$$\nu(A) = \mathbb{E}\left[\mu^L(1, A)\right] = \mathbb{E}\left[\sum_{s \le 1} \mathbf{1}_A(\Delta L_s(\omega))\right]$$

is the Lévy measure of the Lévy process L.

The Lévy-Itô decomposition

Theorem

Consider a triple (b, c, ν) *of an inf. divisible law. Then there exists a prob. space and 4 independent Lévy processes* $L^{(1)}$, $L^{(2)}$, $L^{(3)}$ *and* $L^{(4)}$ *such that*

$$L = L^{(1)} + L^{(2)} + L^{(3)} + L^{(4)}$$

is a Lévy process with characteristic triplet (b, c, ν) *and*

$$L_t^{(1)} = bt; \quad L_t^{(2)} = \sqrt{c} W_t,$$

$$L_t^{(3)} = \int_0^t \int_{|x| \geq 1} x \mu^L (ds, dx),$$

$$L_t^{(4)} = \int_0^t \int_{|x| < 1} x \left(\mu^L - \nu^L \right) (ds, dx).$$

The Lévy measure, paths and moment properties

- ν satisfies $\nu(\{0\}) = 0$, $\int_{\mathbb{R}} \left(1 \wedge x^2\right) \nu(dx) < \infty$ and describes the expected number of jumps at a certain level in a time interval of size 1.
- If $\nu(\{\mathbb{R}\}) = \infty$ then infinitely many jumps occur (small jumps). The Lévy process has infinite activity.
- If $\nu(\{\mathbb{R}\}) < \infty$ then a.a. paths have a finite number of jumps. The Lévy process has finite activity.
- Let L be a Lévy process with triplet (b, c, ν). If $c = 0$ and $\int_{|x| \leq 1} |x| \nu(dx) < \infty$ then a.a. paths have finite variation. If $c \neq 0$ or $\int_{|x| \leq 1} |x| \nu(dx) = \infty$ then a.a. paths have infinite variation.

The Lévy measure, paths and moment properties

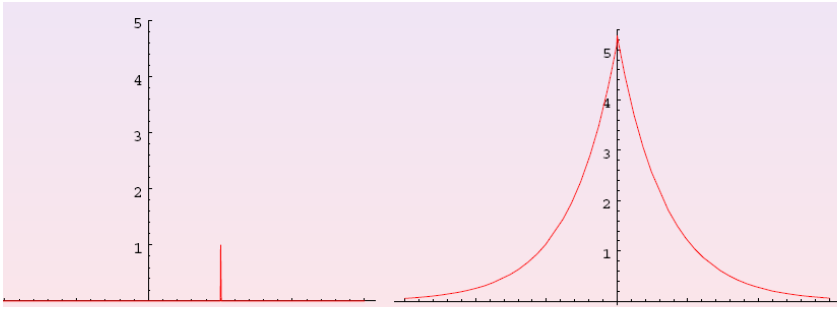


Figure: The Lévy measure of the Poisson and of a compound Poisson process

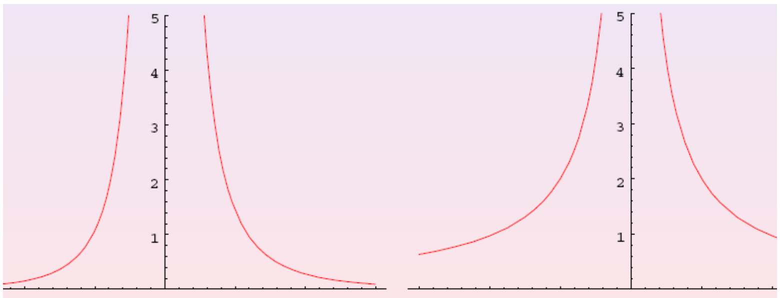


Figure: The Lévy measure of a NIG and an α-stable process

The Lévy-Itô decomposition

The Lévy measure, paths and moment properties

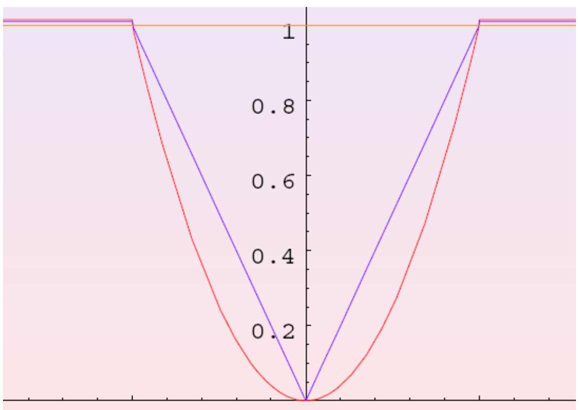


Figure: $|x|^2 \wedge 1$ (red). $|x| \wedge 1$ (blue)

The Lévy measure, paths and moment properties

- The path variation properties depend on the small jumps (and Brownian motion).
- The activity depends on all the jumps.
- The moment properties depend on the big jumps.
- The finiteness of the moments of a Lévy processes is related to the finiteness of an integral over the Lévy measure (considering only big jumps).
- L_t has finite moment of order p iff $\int_{|x|\geq 1} |x|^p \, \nu(dx) < \infty$.
- L_t has finite exponential moment of order p $\left(\text{i.e. } \mathbb{E}\left[e^{pL_t}\right] < \infty\right)$ iff $\int_{|x|\geq 1} e^{px} \nu(dx) < \infty$.

Models

- Subordinator: it is an a.s. increasing (in t) Lévy process.
- A Lévy process is a subordinator if $\nu(-\infty, 0) = 0$, $c = 0$, $\int_{(0,1)} x\nu(dx) < \infty$ and $b \geq 0$.
- The characteristic exponent is

$$\psi(u) = ibu + \int_0^\infty \left(e^{iux} - 1\right) \nu(dx)$$

- The Poisson process is a subordinator.

Asset price models

In the risk neutral-world, the asset price process is

$$S_t = S_0 \exp(L_t), \qquad 0 \leq t \leq T$$

- L_t is a Lévy process with triplet $\left(\overline{b}, \overline{c}, \overline{\nu}\right)$.and canonical decomposition

$$L_t = \overline{b}t + \sqrt{\overline{c}}\overline{W}_t + \int_0^t \int_{\mathbb{R}} x\left(\mu^L - \overline{\nu}^L\right)(ds, dx)$$

with

$$\overline{b} = r - q - \frac{\overline{c}}{2} - \int_{\mathbb{R}} (e^x - 1 - x)\,\overline{\nu}(dx)$$

Option pricing

- Transform methods
- PDIE's methods
- Monte-Carlo methods

Models

- Black-Scholes model: $L_1 \sim N\left(\mu, \sigma^2\right)$. The Lévy triplet is $\left(\mu, \sigma^2, 0\right)$ and $L_t = \mu t + \sigma W_t$.
- Merton (jump-diffusion) model: $L_t = \mu t + \sigma W_t + \sum_{k=1}^{N_t} J_k$, with $J_k \sim N\left(\mu_J, \sigma_J^2\right)$ (with density f_J). The Lévy triplet is $\left(\mu, \sigma^2, \lambda \times f_J\right)$.

Some models

Models

- The Variance Gamma process: It has a characteristic function given by a Variance Gamma distribution $VG(\sigma, \nu, \theta)$ and:

$$\phi_u(t) = \left(1 - iu\theta v + \frac{1}{2}\sigma^2 \nu u^2\right)^{-\frac{t}{\nu}}$$

  It has Lévy triplet $(\gamma, 0, \nu_{VG}(dx))$.
- The Variance Gamma process can be defined as a time-changed Brownian motion with drift:

$$L_t = \theta G_t + \sigma W_{G_t},$$

  where G is a Gamma process with two appropriate parameters.
- Normal inverse Gaussian model (NIG)
- CGMY model
- Meixner model
- etc...

- Cont, R. and P. Tankov (2003). Financial modelling with jump processes - See Chapter 1.
- Papantaloleon, A (2008). An Introduction to Lévy Processes with Applications in Finance. arXiv:0804.0482v2. - See sections 1-3 and 18.
- Schoutens (2003). Lévy Processes in finance. See chapters 3 and 4.