Models in Finance - Class 7

Master in Actuarial Science

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ISEG

Cross-sectional and longitudinal properties

- Two ways of looking at the properties of time series models. For a given quantity, we can consider a table: in each row is one simulation and each column corresponds to a future projetion date and all simulation start from the same starting position.

simulation i	$t=0$	$t=1$	$t=2$	$t=3$	$\cdots$	$t=T$
$i=1$	2.1	3.2	4.1	5.6	$\cdots$	2.5
$i=2$	2.1	3.1	4.2	5.4	$\cdots$	2.7
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$i=N$	2.1	2.8	4.4	5.3	$\cdots$	2.8

Cross-sectional and longitudinal properties

- Cross-sectional property: fixes time horizon and looks at the distribution over all the simulations (look at a column)
- Example: distribution of inflation next year? Implicitly, the distribution on a column is conditional on the past information that is built on initial conditions.
- If the initial conditions change, the implied cross-sectional distribution will also change.
- Therefore, cross-sectional properties are difficult to validate from past data (each year of past history started from a different set of conditions).
- However, prices of derivatives today reflect market views of a cross-sectional distribution and cross-sectional information can sometimes be deduced from market prices of derivatives (example: implied volatility).

Cross-sectional and longitudinal properties

- Longitudinal property: Fixes a simulation and looks at a statistic sampled repeatedly from that simulation over a long period of time (look at a row).
- Example: fix the second simulation ($i = 2$) and fit a distribution to the sample rates of inflation projected for the next 1000 years.
- Example: look at $I(2)$ of the 2nd simulation ($i = 2$). If we want to study the distribution of $I(2)$, we can first sample value is 4.2, then the following number in the row is 5.4 which is the 2nd value in our sample (but with starting point 3.1) and so on... we will obtain the longitudinal distribution of $I(2)$ from row 2.
- Ergodic theorems: for some models, the longitudinal distribution will converge to a limiting distribution when the time horizon goes to $+\infty$ and this limiting distribution is common to all simulations. Such convergence results are known as "ergodic theorems". The limiting distribution is known as an "ergodic distribution".

Cross-sectional and longitudinal properties

- Unlike cross-sectional properties, longitudinal properties do not reflect market conditions at a particular date. Instead, they reflect an "average"over all likely future economic conditions. Most statistical properties computed from historical data are longitudinal properties.
- In the lognormal model, asset resturns are independent across years and also (as for any model) across simulations. Therefore, for the lognormal model, the cross-sectional and longitudinal properties coincide.
- Any evidence that cross-sectional and longitudinal values of returns are not equal is an evidence against the lognormal model.

Cross-sectional and longitudinal properties

- In general, a cross-sectional property should not be used when a longitudinal one is required and vice-versa.
- Example: as input to an option pricing model, one should use the implied volatility (a cross sectional property) and not the historical volatility (a longitudinal property: is an estimated longitudinal standard deviation).
- In the Wilkie model (an autoregressive model) it is not valid to equate cross-sectional and longitudinal properties.

Autoregressive models

- Random walk processes are expected to grow arbitrarily large with time (if they have a positive drift) and the log-share price process and the share price process are non-stationary.
- Many financial variable should not behave like this. Example: the interest rates...
- The returns are also independent in the lognormal model but this should not be expected for all financial variables (example: interest rates). We expect that some mean-reverting "force"to pull interest rates back to some "normal range". Similar expectations for the annual rate of growth for dividends or prices or inflation. These quantities should not be independent from one year to the next: they tend to form clusters, i.e. the models should be autoregressive.

Autoregressive models

- General ARMA(p, q) model X_t is defined by:

$$X_t = \mu + \alpha_1 (X_{t-1} - \mu) + \cdots + \alpha_p (X_{t-p} - \mu) + e_t + \beta_1 e_{t-1} + \cdots + \beta_q e_{t-q},$$

  where e_t is a zero-mean white noise process.
- AR(1) process:

$$X_t = \mu + \alpha (X_{t-1} - \mu) + e_t.$$

  This process is stationary in the long-run if and only if $|\alpha| < 1$. In that case, we can rearrange the eq. to show that the process is mean-reverting:

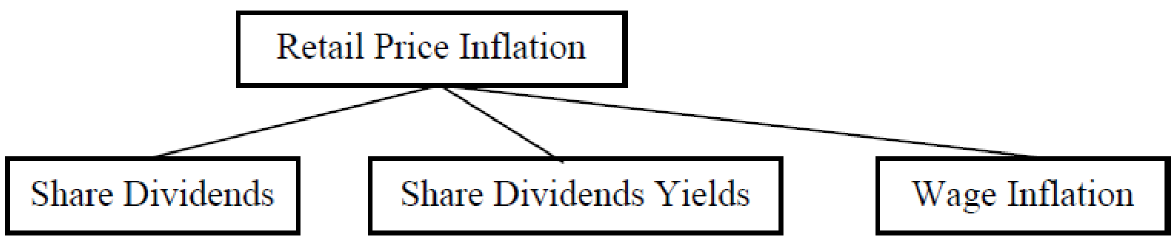
$$X_t - \mu = \alpha (X_{t-1} - \mu) + e_t.$$

Autoregressive models

- Exercise: If $\alpha = 1$, what would be the model? Does it has reversion to the mean properties? And what if $\alpha > 1$?
- Exercise: Which of the following variables are likely to be mean-reverting:
  (a) Money supply
  (b) Monetary growth rates
  (c) Actuarial salaries

Wilkie model

- The Wilkie model (paper by David Wilkie: "More on a Stochastic Asset Model for Actuarial Use", BAJ Volume 1, pages 777–964) is a multivariate time series model designed explicitly for long term applications. It is used to simulate the future values of economic variables in each future year.
- Structure (cascade model)



etc... (subset of the full model)

Wilkie model

- Other variables of the model: Dividend Growth, Share Prices, Interest Rates (short- and long-term), Property Returns, Currency Exchange Rates, etc...
- The Wilkie model can also be recast as a single multivariate model based on a vector autoregressive moving average (VARMA) process.
- The original form was decided upon using a combination of high level economic expectations, statistical analysis and the requirement to keep the model parsimonious (avoid "too many" parameters).
- There is no strong economic reason for the model to have the particular form it was originally published. Analysis using different data sets may indicate that the best statistical model has a different form.

Wilkie model

- Key variables of the model:
  Force of inflation during year t: $I(t)$ (it is derived from movements in the Retail Price Index $Q(t)$ so that $I(t) = Q(t) - Q(t-1)$)
  The equity dividend yield at the end of year t: $Y(t)$
  The force of dividend growth during year t: $K(t)$
  The Real yield on perpetual index-linked bonds at the end of year t: $R(t)$
- $Y(t)$ and $R(t)$ are positive. That is why is important to model $\log Y(t)$ and $\log R(t)$. They are modelled by ARMA processes.

Wilkie model

- In the original Wilkie model, the variables $I(t)$, $\log Y(t)$, $K(t)$ and $\log R(t)$ follow (variations on) mean-reverting processes.
- Wilkie also considers a moving average, $DM(t)$, of "smoothed inflation"that arises as an intermediate step in modelling dividend growth and is calculated by averaging the values of $I(t)$.
- Note: The relationship between the force of inflation $I(t)$ and the rate of inflation $Inf(t)$ is $I(t) = \log(1 + Inf(t))$

Wilkie model

- Wilkie's notation:
  MU (mean)
  SD (standard deviation)
  A (autoregressive parameter)
  E (residual) corresponding to a AR(1) model.
- The model

$$x_t = \mu + a(x_{t-1} - \mu) + e_t$$

corresponds in Wilkie notation to:

$$X(t) = XMU + XA(X(t-1) - XMU) + XE(t)$$

Wilkie model

- Wilkie denotes unit normal r.v. as $Z(t)$ so that $E(t) = SD \times Z(t)$ and $XE(t) = XSD.XZ(t)$.
- Transfer function parameters are W (or D, where a moving average effect is being denoted). Transfer functions are used to express relationships between two different variables. The "W"parameters reflect weightings that are applied to one variable when calculating the value of another.
- M is used to denote an exponentially weighted moving average of previous values and N denotes part of a variable that has zero mean.
- In each case, the parameter symbol is preceded by the single letter denoting which variable is being considered.

Wilkie model

- Example: $RZ(t)$ represents the unit normal variable used in the index-linked real yield model.
- The updating equations are the time series relations that define the model: how each variable is updated from one year to the next.
- Updating eq. for inflation:

$$I(t) = QMU + QA\left[I(t-1) - QMU\right] + QE(t),$$

where:

- $QE(t) = QSD.QZ(t)$,
- QMU, QA and QSD are parameters to be estimated.
- $QZ(t)$ is a series of i.i.d. random standard normal variables, also called "innovations".

Wilkie model

- Structure of equation: this year's value = long-run mean + $QA \times$ (last year's value − long-run mean) + "a shock to the system".
- Exercise: What value or range might be appropriate for QA?
- Properties: Inflation mean-reverts to a constant long-term value QMU, the value of inflation in any future year follows a normal distribution, the model allows inflation to take negative values (deflation).