Markov Processes, martingales and Poisson integration

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Markov processes

- Let $(\Omega, \mathcal{F}, P)$ be a filtered probability space with filtration $(\mathcal{F}_t, t \geq 0)$.
- A stochastic process $X = (X(t), t \geq 0)$ is adapted to the $(\mathcal{F}_t, t \geq 0)$ if each $X(t)$ is $\mathcal{F}_t$-measurable
- Any process X is adapted to its natural filtration $\mathcal{F}_t^X := \sigma \{X(s), s \leq t\}$.

Definition

An adapted process X is a Markov process if for all measurable bounded function f, we have (for $s \leq t$)

$$E[f(X(t)) | \mathcal{F}_s] = E[f(X(t)) | X(s)] \quad \text{a.s.}$$

- Markov process: "past and future are independent, given the present".
- Transition probabilities of a Markov process: $p_{s,t}(x, A) = P[X(t) \in A | X(s) = x]$

Markov processes

Theorem

If X is an adapted Lévy process where each $X(t)$ has law q_t, then it is a Markov process with transition probabilities:

$$p_{s,t}(x,A) = q_{t-s}(A-x). \tag{1}$$

Proof: By the stationarity of increments,

$$\begin{aligned} E[f(X(t))|\mathcal{F}_s] &= E[f(X(s)+X(t)-X(s))|\mathcal{F}_s] \\ &= \int_{\mathbb{R}^d} f(X(s)+y)\, q_{t-s}(dy). \end{aligned}$$

Hence,

$$E[f(X(t))|\mathcal{F}_s] = E[f(X(t))|X_s]$$

and the transition probabilities are obtained for $f = \chi_A$ and $p_{s,t}(x,A) = \int_{\mathbb{R}^d} \mathbf{1}_A(x+y)\, q_{t-s}(dy) = q_{t-s}(A-x)$.■

- Lévy processes are completely characterized by condition (1): they are the only Markov processes which are homogenous in space and time.

Martingales

Definition

The process X is a martingale if X is adapted to $(\mathcal{F}_t, t \geq 0)$, $E[|X(t)|] < \infty$ for all $t \geq 0$ and

$$E[X(t)|\mathcal{F}_s] = X_s \quad \text{a.s} \quad \text{for all } s < t.$$

Theorem

An adapted Lévy process with finite first moment and zero mean is a martingale (with respect to its natural filtration)

Proof: X adapted, $E[|X(t)|] < \infty$ for all $t \geq 0$ and

$$\begin{aligned} E[X(t)|\mathcal{F}_s] &= E[X(s)+X(t)-X(s)|\mathcal{F}_s] \\ &= X(s) + E[X(t)-X(s)] = X(s). \end{aligned}$$

Martingales

Examples of Lévy processes that are also martingales:

- 1. $\sigma B(t)$, $B(t)$ d-dim. BM and σ an $r \times d$ matrix.
  2. $\widetilde{N}(t) = N(t) - \lambda t$ - compensated Poisson process
  3. $\exp\{iuX(t) - t\eta(u)\}$ where $u \in \mathbb{R}$ is fixed and X is a Lévy process with Lévy symbol η.
  4. $|\sigma B(t)|^2 - trace(A)\,t$, with $A = \sigma^T \sigma$
  5. $\left[\widetilde{N}(t)\right]^2 - \lambda t$
- Exercise: Show that $\exp\{iuX(t) - t\eta(u)\}$ is a martingale.

Càdlág paths

- $f : \mathbb{R}^+ \to \mathbb{R}$ is a càdlàg function if is "continue à droite et limité à gauche" - right continuous with left limits.
- Notation: $f(t-) := \lim_{s \uparrow t} f(s)$ and $\Delta f(t) := f(t) - f(t-)$.
- If f is càdlàg then $\#\{0 \leq t \leq T : \Delta f(t) \neq 0\}$ is at most countable.
- If the filtration satisfies the "usual hypothesis" then every Lévy process has a càdlàg modification which is itself a Lévy process (proof: theorem 2.1.8, pag 87 - Applebaum).
- Usual hypothesis for $(\mathcal{F}_t, t \geq 0)$:
  1. (completeness): $\mathcal{F}_0$ contains all sets of P-measure 0.
  2. (right continuity): $\mathcal{F}_t = \mathcal{F}_{t+}$ where $\mathcal{F}_{t+} = \cap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon}$ (allows an "infinitesimal peek" into the future).

Assumptions

From now on, we will allways assume that:

- $(\Omega, \mathcal{F}, P)$ will be a fixed filtered probability space with a filtration $(\mathcal{F}_t, t \geq 0)$ which satisfies the "usual hypotheses".
- Every Lévy process X will be assumed to be $\mathcal{F}_t$-adapted and with càdlàg sample paths.
- $X(t) - X(s)$ is independent of $\mathcal{F}_s$ for all $s < t$.
- Note: given two processes $(X(t), t \geq 0)$ and $(Y(t), t \geq 0)$ we say that Y is a modification of X if, for each $t \geq 0$, $P[X(t) \neq Y(t)] = 0$. As a consequence X and Y have the same finite dimensional distributions.

The jumps of a Lévy process

- The jump process ΔX associated to X is defined by

$$\Delta X(t) = X(t) - X(t-).$$

Theorem

If N is an increasing, integer-valued Lévy process such that $\Delta N(t)$ takes values in $\{0, 1\}$ then N is a Poisson process.

Proof: see Applebaum (2005). Lectures on Lévy Processes, Lecture 2, page 2.

Lemma

If X is a Lévy process, then for fixed $t > 0$, $\Delta X(t) = 0$ a.s.

The jumps of a Lévy process

Proof:

- Let $(t(n); n \in N)$ be a sequence in $\mathbb{R}^+$ with $t(n) \uparrow t$ as $n \to \infty$.
- X has càdlàg paths $\Longrightarrow \lim_{n\to\infty} X(t(n)) = X(t-)$.
- By the stochastic continuity condition (in the Lévy process definition) $\Longrightarrow X(t(n))$ converges in probability to $X(t)$, and so has a subsequence which converges a.s to $X(t)$. Then, by the uniqueness of the limits $X(t) = X(t-)$ (a.s.) and $\Delta X(t) = 0$ (a.s.).■

The jumps of a Lévy process

- Analytic difficulty in manipulating Lèvy processes has to do with the fact that is possible to have:

$$\sum_{0\leq s\leq t} |\Delta X(s)| = \infty \quad \text{a.s.}$$

- To overcome this difficulties, we will use the fact that always:

$$\sum_{0\leq s\leq t} |\Delta X(s)|^2 < \infty \quad \text{a.s.}$$

- In order to count jumps of specified size, define (for a set $A \in \mathcal{B}\left(\mathbb{R}^d - \{0\}\right)$):

$$\begin{aligned} N(t,A) &= \#\{0 \leq s \leq t : \Delta X(s) \in A\} \\ &= \sum_{0\leq s\leq t} \mathbf{1}_A(\Delta X(s)) \end{aligned}$$

- For each $\omega \in \Omega$, $t \geq 0$, the map $A \to N(t,A)$ is a counting measure on $\mathcal{B}\left(\mathbb{R}^d - \{0\}\right)$. (Note: $\mathcal{B}\left(\mathbb{R}^d - \{0\}\right)$ is the $\sigma-$algebra of Borelian measurable sets in $\mathbb{R}^d - \{0\}$)

The jumps of a Lévy process

- Then
$$E\left[N(t,A)\right] = \int N(t,A)(\omega)\, dP(\omega)$$
is a measure on $\mathcal{B}\left(\mathbb{R}^d - \{0\}\right)$.
- Notation: $\mu(\cdot) = E\left[N(1,\cdot)\right]$ is a measure on $\mathcal{B}\left(\mathbb{R}^d - \{0\}\right)$ called the intensity measure (considers the mean number of jumps until time 1).
- We say that $A \in \mathcal{B}\left(\mathbb{R}^d - \{0\}\right)$ is bounded below if $0 \notin \overline{A}$ (note: $\overline{A}$ is the closure of $A =$all points in A plus the limit points of A).

Lemma

If A is bounded below then $N(t,A) < \infty$ a.s. for all $t \geq 0$.

Markov processes, Martingales and Poisson integration

The jumps of a Lévy process

Sketch of the Proof: Define the stopping times $(T_n^A, n \in \mathbb{N})$ by
$T_1^A = \inf\{t > 0 : \Delta X(t) \in A\}$ and $T_n^A = \inf\left\{t > T_{n-1}^A : \Delta X(t) \in A\right\}$
X has càdlàg paths $\Longrightarrow T_1^A > 0$ a.s. and $\lim_{n\to\infty} T_n^A = \infty$ a.s. Otherwise, the set of all jumps in A would have an accumulation point, and this is not possible if X is càdlàg (see the proof of Theorem 2.8.1 in appendix 2.8 of Applebaum).
Moreover,
$$N(t,A) = \sum_{n\in\mathbb{N}} \mathbf{1}_{\left\{T_n^A \leq t\right\}} < \infty \quad \text{a.s.}$$

The jumps of a Lévy process

- If A fails to be bounded below, then the Lemma may no longer hold, because of the accumulation of large numbers of small jumps.

Theorem

1. If A is bounded below, then the process $(N(t, A), t \geq 0)$ is a Poisson process with intensity $\mu(A)$.
2. If $A_1, \ldots A_m \in \mathcal{B}\left(\mathbb{R}^d - \{0\}\right)$ are disjoint then the r.v. $N(t, A_1), \ldots, N(t, A_m)$ are independent.

Proof: pages 101-103 of Applebaum.

Markov processes, Martingales and Poisson integration

The jumps of a Lévy process

- Consequence: $\mu(A) < \infty$ whenever A is bounded below.
- Main properties of N:
  1. For each t and $\omega \in \Omega$, $N(t, \cdot)(\omega)$ is a counting measure on $\mathcal{B}\left(\mathbb{R}^d - \{0\}\right)$.
  2. For each A bounded below, $(N(t, A), t \geq 0)$ is a Poisson process with intensity $\mu(A) = E[N(1, A)]$.
  3. The compensated $(\widetilde{N}(t, A), t \geq 0)$ is a martingale-valued measure where $\widetilde{N}(t, A) = N(t, A) - t\mu(A)$, for A bounded below, i.e. for fixed A bounded below, $(\widetilde{N}(t, A), t \geq 0)$ is a martingale.

Poisson integration

- Let f be a measurable function from $\mathbb{R}^d$ to $\mathbb{R}^d$ and let A be bounded below. Then we may define the Poisson integral of f as the random finite sum

$$\int_A f(x)\, N(t, dx)(\omega) = \sum_{x \in A} f(x)\, N(t, \{x\})(\omega),$$

where $\{x\}$ are the jump sizes of the process (in A), i.e. $N(t, \{x\}) \neq 0 \iff \Delta X(u) = x$ for some $0 \leq u \leq t$.
- $\int_A f(x)\, N(t, dx)$ is a $\mathbb{R}^d$-valued r.v. and gives rise to a càdlàg stoch. process as we vary t.
- We have also

$$\int_A f(x)\, N(t, dx) = \sum_{0 \leq u \leq t} f(\Delta X(u))\, \mathbf{1}_A(\Delta X(u)).$$

Poisson integration

Theorem

Let A be bounded below. Then:
1. $\left(\int_A f(x)\, N(t, dx), t \geq 0\right)$ *is a compound Poisson process with characteristic function*

$$\exp\left(t \int_{\mathbb{R}^d} \left(e^{i(u,x)} - 1\right) \mu_{f,A}(dx)\right),$$

where $\mu_{f,A}(B) = \mu\left(A \cap f^{-1}(B)\right)$ *for* $B \in \mathcal{B}\left(\mathbb{R}^d\right)$.
2. If $f \in L^1(A, \mu_A)$ *then (*μ_A *is the restriction to A of the measure* μ*):*

$$\mathbb{E}\left[\int_A f(x)\, N(t, dx)\right] = t \int_A f(x)\, \mu(dx).$$

3. If $f \in L^2(A, \mu_A)$ *then*

$$\mathrm{Var}\left(\left|\int_A f(x)\, N(t, dx)\right|\right) = t \int_A |f(x)|^2\, \mu(dx).$$

Poisson integration

Sketch of the proof: 1. Assume $f \in L^1(A, \mu_A)$ and let f be a simple function: $f = \sum_{j=1}^n c_j \mathbf{1}_{A_j}$ (with the A_j's disjoint). Then, by part 2 of the previous theorem, we have that

$$E\left[\exp\left\{i\left(u, \int_A f(x) N(t, dx)\right)\right\}\right] = \prod_{j=1}^n E\left[\exp\left\{i\left(u, \int_A c_j N(t, A_j)\right)\right\}\right]$$

$$= \prod_{j=1}^n \exp\left\{t\left(e^{i\left(u, c_j\right)} - 1\right)\mu(A_j)\right\} = \exp\left\{t\left(e^{i(u, f(x))} - 1\right)\mu(dx)\right\}.$$

For an arbitrary $f \in L^1(A, \mu_A)$, we can find a sequence of simple functions converging to f in L^1 and hence a subsequence which converges to f a.s. Passing to the limit along this subsequence yields the required result. Parts 2. and 3. follow from 1. by differentiation (moments from characteristic function: $E\left[X^k\right] = (-i)^k \Phi^{(k)}(0)$)■

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Poisson integration

- It follows from Theorem - part (2) that a Poisson integral will fail to have a finite mean if $f \notin L^1(A, \mu)$.
- For $f \in L^1(A, \mu_A)$, we define the compensated Poisson integral by

$$\int_A f(x) \widetilde{N}(t, dx) = \int_A f(x) N(t, dx) - t \int_A f(x) \mu(dx).$$

- The process $\left(\int_A f(x) \widetilde{N}(t, dx), t \geq 0\right)$ is a martingale.

Poisson integration

- By the previous theorem, we have that

$$E\left[\exp\left\{i\left(u,\int_A f(x)\widetilde{N}(t,dx)\right)\right\}\right]$$
$$=\exp\left(t\int_{\mathbb{R}^d}\left(e^{i(u,x)}-1-i(u,x)\right)\mu_{f,A}(dx)\right)$$

and if $f\in L^2(A,\mu_A)$ then

$$E\left[\left|\int_A f(x)\widetilde{N}(t,dx)\right|^2\right]=t\int_A|f(x)|^2\mu(dx).$$

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