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TIME CHANGES FOR LÉVY PROCESSES

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The goal of this paper is to consider pure jump Lévy processes of finite variation with an infinite arrival rate of jumps as models for the logarithm of asset prices. These processes may be written as time-changed Brownian motion. We exhibit the explicit time change for each of a wide class of Lévy processes and show that the time change is a weighted price move measure of time. Additionally, we present a number of Lévy processes that are analytically tractable, in their characteristic functions and Lévy densities, and hence are relevant for option pricing.

KEY WORDS: purely discontinuous processes, finite variation processes, Brownian excursions, infinite activity, completely monotone Lévy density

1. INTRODUCTION

Continuity of price processes has served finance as a convenient and powerful assumption, delivering market completeness and unique pricing of derivatives by arbitrage. Such assumptions are critical to the validity of the Black–Scholes (1973) and Merton (1973) option pricing theories and the associated dynamic hedging strategies. The assumption of continuity justifies the Cox, Ross, and Rubinstein (1979) binomial approximation of the process using up and down states that have price shocks tending to zero as one decreases the time grid. We grant that such price processes can accurately represent market prices in economies which instantaneously and continuously equilibrate to information flows that are driven by diffusions or Itô processes. Examples of such equilibrium models in the literature include Duffie and Huang (1985), Dumas (1989), and Detemple and Murthy (1994). We note that the continuity of the resulting price processes is a consequence of the assumed continuous information flows.

We consider instead discontinuous information flows represented by purely discontinuous stochastic processes for the underlying uncertainty. These processes replace the role of Brownian motion in the classical models and the resulting price process can be expressed as the difference between two increasing random processes that account for the upward and downward moves of the market. Thus, our processes are both purely discontinuous and, unlike Brownian motion, of finite variation.

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The possibility of jumps in asset prices has a long history in the finance literature. Merton (1976) considered the addition of a jump component to the classical geometric Brownian motion model for the pricing of options on stocks. In explaining the distribution of returns such models were used as early as Mandelbrot (1963) and Press (1967). The motivation for the use of jumps and the associated heavy-tailed distributions is twofold. First we have the presence of excess kurtosis relative to the normal distribution in the time series of asset returns. This has been very well documented in the finance literature since Fama (1965), and more recently by Eberlein and Keller (1995), for instance. Second, we have the existence of smiles in Black–Scholes implied volatilities which is indicative of heavier tails in the risk neutral distribution. This is documented in Bates (1996) and Bakshi, Cao, and Chen (1997). The importance of these considerations is also being increasingly noted and realized as derivative markets expand to cover new underlying assets like electricity (Eydeland and Geman 1998). However, unlike jump diffusions that basically account for high-activity small price moves using an infinite variation diffusion process and low-activity large moves by an orthogonal pure jump process, we consider the use of infinite-activity finite variation pure jump processes that synthesize the small and large moves by requiring smaller moves to occur at a higher rate than larger moves. This is achieved by requiring the Lévy density to be completely monotone. We note in this regard that many of the classsical jump diffusion models employ a Gaussian density for the jump size and this is not completely monotone. Hence our models constitute a new class even from the perspective of the jump structures of classical jump diffusion models.

Though the processes we suggest are pure jump and of finite total variation, we generalize the results of Clark (1973) and show that they may always be viewed as continuous processes evaluated at a random time change. Clark considered subordinated processes that evaluate geometric Brownian motion at a random time given by a process of homogeneous and independent increments which is itself independent of the Brownian motion. The processes we propose can also be written as Brownian motion evaluated at a random time change or a stochastic clock, but in general the process of the clock need not be one of independent and homogeneous increments, and it need not be independent of the price process. This ability to represent our processes as time-changed Brownian motion is an important and attractive connection between the representations considered here and classical diffusion based models, suggestive of potential differences between economic and calendar time. An important contribution of this paper is the explicit study of the relationship between the time change or stochastic clock and the original price process. We observe that in economies with a high (infinite) activity rate, price continuity does arise, but in an activity-based measure of time, as opposed to calendar time directly.

The outline of the paper is as follows: Section 2 presents arguments in support of the use of purely discontinuous time-changed Brownian motions as the universal candidate class of processes for financial market asset prices. Section 3 considers specific examples relating the time change to the price process. Section 4 concludes.

2. STOCHASTIC TIME CHANGES AND THE PRICE PROCESS

We present here some general arguments supporting the use of purely discontinuous time-changed Brownian motions as models for financial market price processes. We begin with the implications of no arbitrage and then address issues of local uncertainty in time changes. We denote the process for the price of a financial asset by $p(t)$ and will generally be concerned with models for the logarithm of this price.

It is generally agreed that models for $\ln(p(t))$ should be consistent with the absence of arbitrage opportunities. It is known that such processes must be semimartingales (Delbaen and Schachermayer 1994). It is also known (e.g., Monroe 1978) that every semimartingale can be written as a Brownian motion (possibly defined on some adequately extended probability space) evaluated at a random time change. By such a result there exists a Brownian motion $(W(u), u \geq 0)$ and a random time change $T(t)$ where $T(t)$ is an increasing stochastic process[1] such that

$$\ln(p(t)) = \ln(p(0)) + W(T(t)). \tag{2.1}$$

This suggests that return distributions should be normal, not when measured with respect to calendar time, but when measured per unit of what may be termed as an economically relevant measure of time. The search for such a formulation began in earnest with Clark (1973), and has been pursued more recently by Geman and Ané (1996), Ané and Geman (2000), Madan, Carr, and Chang (1998), and Bakshi and Madan (1999). In fact, it is shown in Geman and Ané that one-minute S&P 500 calendar time returns are highly nonnormal, but that returns measured per unit trade are normally distributed. Madan et al. show that Brownian motion time-changed by a gamma process provides a significantly better description of historical asset returns and the risk-neutral return distribution embedded in option prices. In fact, the time-changed process successfully combats the well-known smile effects of Black–Scholes pricing for short maturity S&P 500 index options. Bakshi and Madan (1999) on the other hand use this process to learn about the probabilities of large market moves from small ones. Our investigations here will focus on relating $T(t)$ to the original process for $\ln(p(t))$.

We first consider representations of the type given by equation (2.1) in which the Brownian motion is independent of the time change. Furthermore, to allow for additional generality we consider stochastic time changes applied to continuous Itô processes. From this perspective, let $x(t)$ be an Itô process of the general form

$$x(t) = x(0) + \int_0^t \theta(u)du + \int_0^t \sigma(u)dW(u). \tag{2.2}$$

We consider the representation of the price process as $x(t)$ evaluated at a random time $T(t)$ or

$$\ln(p(t)) = \ln(p(0)) + x(T(t)), \tag{2.3}$$

where $T(t)$ is independent of $(x(u), u \geq 0)$.

Some general properties of the time change $T(t)$ are tied to that of the price process. First, if the log price process is pure jump then since $x(t)$ is continuous it follows that $T(t)$ is a pure jump process and vice versa. Second, we say that the price process represents a high level of activity if there is no interval of time in which prices are constant throughout the time interval. Such high activity processes have an infinite arrival rate of price moves, though of necessity the arrival rate is finite for all moves of a size strictly bounded away from zero. Hence, all but finitely many of the infinitely many price moves are arbitrarily small in size. If the price process is such a high activity process,

[1] More formally, it is required that there exists a filtration $(\mathcal{G}_u)$ with respect to which the process $(W(u), u \geq 0)$ is adapted and, for any given u, its increments from time u onward are independent of $(\mathcal{G}_u)$ and that $T(t)$ is an increasing sequence of stopping times adapted to this filtration.

then this property is also inherited by the time change and it is also a high activity process.

We note now that if the time change $T(t)$ was continuous then under fairly general conditions (specifically when its quadratic characteristic is absolutely continuous with respect to Lebesgue measure) the time change would be an Itô process in its own right and we could write

$$T(t) = T(0) + \int_0^t \zeta(u) du + \int_0^t \eta(u)\, dW(u) \tag{2.4}$$

for some coefficient functions ζ, η. But $T(t)$ is by definition an increasing process and hence we must have that $\eta = 0$ with the consequence that the time change is locally deterministic. If time changes are broadly related to information flows embedded in price processes and there is local uncertainty in these flows then the time change will be a purely discontinuous process. It follows that the price process will be purely discontinuous as well.

3. TIME CHANGES RELATED TO PRICE VARIATION

This section studies in some simple and tractable contexts the relationship of stochastic time changes to the price process. We begin with processes that have a finite arrival rate and then consider high-activity processes with infinite arrival rates. We study these relationships between time changes and price moves in the context of various examples for the price process. Our approach to this problem is to follow the traditional practice pioneered in Black–Scholes (1973) and Merton (1976) whereby one postulates various stochastic processes for the price path, except that we make our assumptions directly on the upward and downward log price movements.

3.1. Compound Poisson Log Price Moves

Finite arrival rate jump processes are not likely to be relevant from an empirical viewpoint when taken in isolation. They are important in the literature as they constitute the jump component of the Merton (1976) jump diffusion model. We consider two cases, the first is the reflected normal as the normal distribution is used in the Merton model, and the second is the exponential that is fundamental to the class of completely monotone jump intensities studied later.

3.2. Reflected Normal Price Shocks

Let the log price upward (and downward) moves be given by independent copies of the compound Poisson process

$$X(t) = \sum_{i=1}^{N(t)} Y_i, \tag{3.1}$$

where $N(t)$ is Poisson process with arrival rate λt and the sequence of the magnitude of log price moves Y_i is independently and identically distributed with a reflected normal density,

$$f(y) = \frac{\sqrt{2}\exp\left(-\frac{y^2}{2\sigma^2}\right)}{\sigma\sqrt{\pi}}, \quad \text{for } y > 0. \tag{3.2}$$

We show that in this example the time change to be applied to Brownian motion to recover the price process turns out to be the total number of price moves.

We start with our representation of the price process

$$\ln(p(t)/p(0)) = X_1(t) - X_2(t), \tag{3.3}$$

where $X_1(t)$, $X_2(t)$ are two independent copies of the process satisfying (3.1).

We then compute the characteristic function $\phi_Y(u)$ of Y

$$\phi_Y(u) = \int_0^\infty \exp(iuy) \frac{\sqrt{2}\exp\left(-\frac{y^2}{2\sigma^2}\right)}{\sigma\sqrt{\pi}} dy; \tag{3.4}$$

hence

$$\mathrm{Re}(\phi_Y(u)) = \exp\left(-\frac{\sigma^2 u^2}{2}\right),$$

in which case by direct computation it follows that

$$\phi_{\ln(p(t)/p(0))}(u) = \exp\left(2\lambda t\left(\exp\left(-\frac{\sigma^2 u^2}{2}\right) - 1\right)\right). \tag{3.5}$$

Equation (3.5), however, is also the characteristic function of $\sigma W(N_1(t) + N_2(t))$, where $W(t)$ is a standard Brownian motion and $N_1(t)$, $N_2(t)$ are two independent copies of a Poisson process with arrival rate λt.

In this example the time change simply counts the number of all the price moves, ignoring the size of these moves. It is interesting to note that the time change of this example is akin to the number of trades time change observed to be relevant by Geman and Ané (1996) in their empirical study of high frequency returns on the S&P 500 futures index.

3.3. Exponential Price Shocks

Suppose now, instead, that the size of price moves has an exponential density

$$f(y) = a\exp(-ay), \quad \text{for } y > 0, \tag{3.6}$$

with mean price move $1/a$. Further suppose that the arrival rate of jumps is $1/a$, so that the Lévy density for the price move is

$$k(x) = \exp(-ax). \tag{3.7}$$

Let $X_1(t)$ and $X_2(t)$ be two independent copies of this exponential compound Poisson process and suppose that log prices evolve in accordance with (3.3). The characteristic function for the log price is now easily evaluated as

$$\phi_{\ln(p(t)/p(0))}(u) = \exp\left(\frac{2t}{a}\left(\frac{a^2}{a^2+u^2} - 1\right)\right). \tag{3.8}$$

To observe this as a time-changed Brownian motion consider the time change given by $T(t)$, a subordinator with Lévy density

$$\tilde{k}(y) = 2a\exp\left(-a^2 y\right), \quad \text{for } y > 0. \tag{3.9}$$

The result follows on evaluating the characteristic function of $\sqrt{2}W(T(t))$ as

$$\begin{aligned} E\big[\exp\big(iu\sqrt{2}W(T(t))\big)\big] &= E\big[\exp\big(-u^2T(t)\big)\big] \\ &= \exp\left(t\int_0^\infty \big(e^{-u^2y}-1\big)2ae^{-a^2y}dy\right) \\ &= \exp\left(2ta\left(\frac{1}{u^2+a^2}-\frac{1}{a^2}\right)\right) \\ &= \exp\left(\frac{2t}{a}\left(\frac{a^2}{a^2+u^2}-1\right)\right). \end{aligned}$$

We observe that a price move of size y arrives at the rate $2\exp(-ay)$ if we account for both up and down moves of this size. To construct the time change we weigh the price moves of different sizes differently; in fact moves of size y amount to $a\exp(a(a-1)y)$ units of time change. For values of a above unity, large moves mean more time passage, and for $a < 1$ it is small moves that add time. Our next example considers two high-activity processes for the price moves.

3.4. Gamma Process Price Moves

Let $U(t)$ be a gamma process with mean and variance rates $\mu_{1,}\nu_1$, while $V(t)$ is an independent gamma process of mean and variance rates μ_2, ν_2. Let $\gamma(t)$ denote the standard gamma process of unit mean and variance rate. One may write $U(t)$ and $V(t)$ in terms of the standard gamma process by

$$U(t) = \frac{\nu_1}{\mu_1}\gamma\left(\frac{\mu_1^2}{\nu_1}t\right), \quad \text{while } V(t) = \frac{\nu_2}{\mu_2}\gamma\left(\frac{\mu_2^2}{\nu_2}t\right).$$

The gamma process itself is a pure jump process with characteristic function

$$\phi_{\gamma(t)}(u) = \left(\frac{1}{1-iu}\right)^t \tag{3.10}$$

and Lévy measure

$$L_\gamma(x)dx = \frac{\exp(-x)}{x}dx \quad \text{for } x > 0 \tag{3.11}$$

that integrates to infinity, and hence we have a process that jumps infinitely often in any interval, or a *high-activity* process.

The price process of equation (3.3) may now be written as

$$\ln(p(t)/p(0)) = \frac{\nu_1}{\mu_1}\gamma\left(\frac{\mu_1^2}{\nu_1}t\right) - \frac{\nu_2}{\mu_2}\gamma\left(\frac{\mu_2^2}{\nu_2}t\right) \tag{3.12}$$

and, letting $\kappa = \mu_1^2/\nu_1 = \mu_2^2/\nu_2$ (by the assumption of a common coefficient of variation), with $\alpha_1 = \nu_1/\mu_1$ and $\alpha_2 = \nu_2/\mu_2$, one may write

$$\ln(p(t)/p(0)) = \gamma_1(t) - \gamma_2(t) = \alpha_1\gamma(\kappa t) - \alpha_2\gamma(\kappa t). \tag{3.13}$$

The characteristic function of $\ln(p(t)/p(0))$ is then easily evaluated as

$$\phi_{\ln(p(t)/p(0))}(u) = \left(\frac{1}{1-i\alpha_1 u}\right)^{\kappa t}\left(\frac{1}{1+i\alpha_2 u}\right)^{\kappa t}$$

$$= \left(\frac{1}{1-i(\alpha_1-\alpha_2)u+\alpha_1\alpha_2 u^2}\right)^{\kappa t}. \tag{3.14}$$

To represent this as a time-changed Brownian motion, consider a Brownian motion with drift θ and volatility σ evaluated at the gamma time, $\gamma_3(t) = \gamma(\kappa t)$. Hence, let

$$Y(t) = \theta\gamma_3(t) + \sigma W(\gamma_3(t)),$$

where $W(t)$ is a standard Brownian motion. The characteristic function of Y conditional on the gamma time is

$$\phi_{Y(t)|\gamma_3(t)}(u) = \exp\left(iu\theta\gamma_3(t) - \frac{\sigma^2u^2}{2}\gamma_3(t)\right) \tag{3.15}$$

and the characteristic function of Y is

$$\phi_Y(u) = \left(\frac{1}{1-iu\theta+\frac{\sigma^2u^2}{2}}\right)^{\kappa t}. \tag{3.16}$$

A comparison of (3.16) and (3.14) shows that with $\theta = \alpha_1 - \alpha_2$ and $\sigma = \sqrt{2\alpha_1\alpha_2}$, the price process may be expressed in probability law as a Brownian motion with drift evaluated at a gamma time, $\gamma_3(t)$.

The time change $\gamma_3(t)$ may be related to the individual gamma processes that were differenced to obtain the price process on noting that

$$\gamma_3(t) = \frac{\mu_1}{2\nu_1}\gamma_1(t) + \frac{\mu_2}{2\nu_2}\gamma_2(t) \tag{3.17}$$

Hence the time change is a scaled variation of the original price process. The up and down price moves are scaled by their mean variance ratios and then summed to compute the passage of time.

A different view on the relationship between price moves and the time change may be obtained by noting that since we are dealing with Lévy processes we need to establish the equivalence in probability law for only one value of t. Hence one may consider $\alpha_1\gamma(\kappa t) - \alpha_2\gamma(\kappa t)$ at $t = (2\kappa)^{-1}$, in which case

$$\ln\left(\frac{p(\frac{1}{2\kappa})}{p(0)}\right) = \alpha_1\gamma(1/2) - \alpha_2\gamma(1/2). \tag{3.18}$$

It is easily verified that the probability law of $2\gamma(1/2)$ is that of the square of a standard normal variate and hence that

$$2\ln\left(\frac{p(\frac{1}{2\kappa})}{p(0)}\right) = \alpha_1 B - \alpha_2 S \tag{3.19}$$

where $B = N_d^2$, $S = N_s^2$ and N_d, N_s are two independent normal random variables of zero mean and unit variance. It is shown in the Appendix that the time change at this same time point is

(3.20) $$\gamma_3(t) = \left(\sqrt{B} - \sqrt{S}\right)^2.$$

Up and down price moves are squared normal variates and the time change is the square of the difference of these normal variates.

3.5. Upward and Downward Price Moves as Completely Monotone Lévy Processes

Mathematically an important structural property of Lévy densities is that of monotonicity. From an economic standpoint one expects that jumps of larger sizes have a lower arrival rate than smaller jumps. This property amounts to asserting for differentiable densities that the derivative is negative for positive jumps and positive for negative jumps. Modeling the negative jumps symmetrically with the positive ones, we restrict the discussion to the Lévy density for the positive jumps.

The property of monotonicity may be strengthened to complete monotonicity by requiring derivatives of the same order to have the same sign and be alternating in sign. This is a useful structural restriction that links analytically the arrival rate of small and large price moves. By Bernstein's theorem all completely monotone Lévy densities are given by the Laplace transform of positive measures $\rho(da)$ on the positive half line and

(3.21) $$k(y) = \int_0^\infty e^{-ay}\rho(da).$$

The gamma process of Section 4.2 is the special case when $k(y) = e^{-y}/y$ as defined in equation (3.11), and this is a completely monotone density. As in the gamma process case, suppose the downward moves $V(t)$ are given by an independent copy of the same Lévy process. The log price process of equation (3.3) now has the form

(3.22) $$\ln(p(t)/p(0)) = U(t) - V(t).$$

It is shown in the Appendix, following the analysis of Leblanc (1997), Knight (1981), and Kotani and Watanabe (1982) that the Lévy density of the time change is given by $k_3(y)$,

(3.23) $$k_3(y) = \int \rho(da)2ae^{-a^2y}.$$

The time change Lévy density given explicitly by (3.23) generalizes to the completely monotone class of densities the result we demonstrated earlier for the exponential Lévy density. The time change is therefore a weighted average of price moves with weight function $a\exp(a(a-1)y)$.

The literature relating price changes to economic activity (Tauchen and Pitts 1983; Karpoff 1987; Gallant, Rossi, and Tauchen 1992; and Jones, Kaul, and Lipson 1994) has focused on volume and the number of trades as the relevant measure of economic activity. The analysis of this section indicates that the time measure incorporates both the number and size of orders as the relevant idea is price impact. Other empirical research on time changes has considered cumulated volatility as a measure of time (Dacoronga et al. 1990) that is generally more appropriate for continuous processes by the result of Dubins and Schwarz (1965).

3.6. Completely Monotone Extensions of the Gamma Process

The Lévy measure of the gamma process is given by equation (3.11). Vershik and Yor (1995) consider this density as the limit of exponentially tilted stable laws with index a, as a tends to zero. The exponentially tilted stable law is an interesting extension in its own right and it introduces an important parameter capable of parameterically differentiating between processes of finite/infinite activity and finite/infinite variation (we refer the reader to Carr et al. (2000) for further details).

3.6.1. *The CGMY Model.* Consider for the price moves x, a Lévy density of the form

$$k_{CGMY}(x) = \begin{cases} \frac{C\exp(-Mx)}{x^{Y+1}} & x > 0 \\ \frac{C\exp(-G|x|)}{|x|^{Y+1}} & x < 0 \end{cases},$$

where $0 \le Y < 1$, for a process with an infinite arrival rate of jumps and a process of finite variation. For values of Y greater than zero the process enhances the arrival rate of small jumps and dampens the arrival rate of large jumps. The case of the gamma process is obtained on setting $Y = 0$. The characteristic function for the upward motion of the log price $X_{CMY}(t)$ is given in closed form by

$$\begin{aligned} E[\exp(iuX_{CMY}(t))] &= \exp\left(t\int_0^\infty (e^{iux}-1)\frac{C\exp(-Mx)}{x^{Y+1}}dx\right) \\ &= \exp\begin{pmatrix} \frac{-tC}{Yx^Y}\left(e^{-(M-iu)x} - e^{-Mx}\right)\Big|_0^\infty + \\ tC\int_0^\infty \frac{x^{-Y}}{Y}(Me^{-Mx} - (M-iu)e^{-(M-iu)x}) \end{pmatrix} \\ &= \exp\left(tC\frac{[\Gamma(1-Y)M^Y - \Gamma(1-Y)(M-iu)^Y]}{Y}\right). \end{aligned} \tag{3.24}$$

The resulting characteristic function for the process X_{CGMY} is

$$E[\exp(iuX_{CGMY}(t))] = \exp\left(tC\Gamma(-Y)\left[(M-iu)^Y - M^Y + (G+iu)^Y - G^Y\right]\right).$$

It is shown in Carr et al. (1999) that different values for the parameters C, G, M, Y lead to different structural properties for the price process. These properties are empirically investigated on a database of equity indices and individual stocks.

3.7. Brownian Excursions Models for the Price Process

We now consider another and equivalent representation of the asset price process that is valid for price processes with completely monotone Lévy densities, a consequence of Krein's theory (Kotani and Watanabe 1982). In this equivalent formulation price changes are related to the excursions of Brownian motion by a measure of the force of the excursion. We introduce the idea of a *force function* that measures the force of an excursion by the integral of its absolute value over an excursion. We show by examples that simple force functions are associated with fairly complex Lévy densities, while the force function for the gamma process is to our knowledge still unknown. Hence the two approaches complement each other and provide a wider class of interesting models.

Unlike the examples of the earlier sections, the time change and the Brownian motion being time changed are no longer independent in the examples of this section. Here the time change represents cumulated volatility, where the latter depends on the Brownian motion itself.

The force function of the price process, denoted $f(x)$, has the property that $f(x) > 0$ for $x > 0$ and $f(x) < 0$ for $x < 0$. Let $W(t)$ be a standard Brownian motion and let $\sigma(t)$ be the inverse local time of this Brownian motion at zero. We note that $W(\sigma(t)) = 0$. We write the price process as

$$\ln(p(t)/p(0)) = \int_0^{\sigma(t)} ds f(W(s)). \tag{3.25}$$

We recall that inverse local time is an increasing pure jump process with Lévy density $L(x)$,

$$L(x) = \frac{1}{\sqrt{2\pi} x^{3/2}}, \ x > 0.$$

Hence the price process (3.25) is a pure jump process with the size of the price move being

$$\int_{\sigma(t_-)}^{\sigma(t)} f(W(s))ds$$

or the integral of the force function over an excursion of the Brownian motion.

The price process (3.25) includes Lévy processes of infinite variation but finite quadratic variation. To obtain processes of bounded variation one must restrict the class of force functions to those that meet the integrability condition

$$\int_{-K}^{K} dx|f(x)| < \infty$$

for all K. Under this integrability condition (which essentially prevents infinite force at zero), one may write the price process as the difference of two independent increasing processes by

$$\ln(p(t)/p(0)) = \int_0^{\sigma(t)} ds f^+(W(s)) - \int_0^{\sigma(t)} ds f^-(W(s)), \tag{3.26}$$

where $f^+(x) = f(x)\mathbf{1}_{(x\geqq 0)}$; $f^-(x) = f(x)\mathbf{1}_{(x\leqq 0)}$. We now model the price increases by $\int_0^{\sigma(t)} ds f^+(W(s))$ and the decreases by $\int_0^{\sigma(t)} ds f^-(W(s))$.

Some interesting facts about the Lévy measure for the price process may be inferred directly from the structure of the force function. For example, one may show that if the force function is zero in an interval around zero, then only excursions of a certain length away from zero contribute to a price movement, and one may infer that the process is one of finite activity with an integrable Lévy measure.[2] Similarly one may show that if the force function is strictly positive in absolute value in an open interval around zero then the process is one of infinite activity with a Lévy measure that has an infinite integral.

[2] This property can be formally proved by integrating the force function with respect to the Itô measure for the height m of the excursion (see Yor 1995, p. 68) which is $(1/m^2)dm$.

The price process of equation (3.25) may once again be expressed as time-changed Brownian motion. Define $\psi(y)$ by $\psi'(y) = f(y)$ and $\psi(0) = 0$; let

$$F(x) = \int_0^x \psi(y)\,dy. \tag{3.27}$$

Itô's lemma may now be applied to $F(W(t))$ in the following way:

$$0 = F(W(\sigma(t))) = \int_0^{\sigma(t)} \psi(W(s))\,dW(s) + \frac{1}{2}\int_0^{\sigma(t)} f(W(s))\,ds \tag{3.28}$$

It follows from equation (3.25) substituted in (3.28) that

$$\ln(p(t)/p(0)) = -2\int_0^{\sigma(t)} \psi(W(s))\,dW(s). \tag{3.29}$$

Hence the probability law of $\ln(p(t)/p(0))$ is that of a Brownian motion β evaluated at $4\int_0^{\sigma(t)} \psi^2(W(s))\,ds$. In this example the Brownian motion and the time change are not independent processes. We note that the time change cumulates the instantaneous volatility of the price process, and observe that this volatility depends on the path of the Brownian motion.

3.8. Characteristic Functions for Price Processes Based on Brownian Excursions

For the statistical evaluation and identification of price processes from data on prices of financial assets including options, the characteristic function of the log price relative is a useful and fundamental construct as noted in Bakshi and Madan (2000) and Carr and Madan (1998). We present here an algorithm for evaluating the characteristic function for the Brownian excursion model presented in Section 4.6, based on the results of Revuz and Yor (1994).

Focusing on the positive and negative movements separately, we seek the Laplace transform of upward moves that is given by Revuz and Yor (1994); see also Bertoin (1999):

$$E\left[\exp\left(-\lambda\int_0^{\sigma(t)} f^+(W(s))\,ds\right)\right] = \exp(t\psi'(0^+)/2), \tag{3.30}$$

where $\psi(x)$ is the unique positive decreasing solution to the Sturm–Liouville equation

$$\frac{1}{2}\psi''(x) = \lambda f^+(x) \tag{3.31}$$

subject to the boundary condition $\psi(0) = 1, \psi(\infty) = 0$, when the function f^+ meets the condition $\int_0^\infty xf^+(x)dx = \infty$. Here we present an analysis based on the Ray–Knight theorem that uses methods commonly employed in the economics literature, and initiated by Cox, Ingersoll, and Ross (1985; hereafter CIR).

One may write the process for the price increase as

$$\int_0^{\sigma(t)} f^+(W(s))\,ds = \int_0^\infty dx f^+(x)L^x_{\sigma(t)}(W), \tag{3.32}$$

where $L^x_{\sigma(t)}(W) = Z(x)$ is the local time of the Brownian motion W at x between 0 and $\sigma(t)$, the inverse local time at zero of W. By the Ray–Knight theorem the process $Z(x)$

viewed as a process in the space variable, for fixed t, is a Feller diffusion and is in fact a Bessel-squared process of dimension 0, starting at t, or in finance terms a CIR process (see Geman and Yor 1993), that satisfies the stochastic differential equation

(3.33) $$dZ(x) = 2\sqrt{Z(x)}dB(x); \qquad Z(0) = t,$$

for a Brownian motion $B(x)$ in the space variable. We are therefore equivalently interested in the Laplace transform of $\int_0^\infty dx f^+(x)Z(x)$. We proceed by considering this problem in the familiar way of analyzing term structure models and define

(3.34) $$G(y, Z) = E\left[\exp\left(-\lambda\int_y^\infty f^+(x)Z(x)\,dx\right)|Z(y) = Z\right].$$

The partial differential equation for G may be derived as

$$G_y + 2ZG_{ZZ} - \lambda f^+(y)Z = 0,$$

which must be solved for the boundary condition $G(y, 0) = 1$, and $G(\infty, Z) = 0$.[3] It is well known (see, e.g., Revuz and Yor 1994, p. 424, Thm. 4.7) that the solution in Z is of the form

$$G(y, Z) = \exp(b(y)Z),$$

where the function b satisfies the Ricatti differential equation (also classically obtained in finance for CIR type models of interest rates or stochastic volatility)

$$b' + 2b^2 = \lambda f^+.$$

The Sturm–Liouville equation follows on making the substitution $b = \frac{1}{2}\frac{\psi'}{\psi}$, and the result follows on evaluating $G(0, Z(0)) = G(0, t)$.

We now consider some explicit examples where the characteristic function of the log price relative may be explicitly evaluated. We consider two cases, one with a decreasing force function and the other with an exponentially increasing force function.

3.9. Example with a Diminishing Force Function

Suppose that

$$f^+(x) = \frac{1}{(kx + l)^2}.$$

Let Z_x be the Feller diffusion associated with the local time at x evaluated at the inverse local time at t. As noted earlier $dZ_x = 2\sqrt{Z_x}dB_x$ for a standard Brownian motion $B(x)$, and $Z_0 = t$. Define $\varphi(x) = (kx + l)^{-1}$ and consider $Y_x = \varphi(x)Z_x$. By construction $Y_0 = t/l$, and an application of Itô's lemma shows that

$$dY_x = -\frac{k}{(kx + l)^2}Z_x dx + 2\varphi(x)\sqrt{Z(x)}dB(x).$$

Writing the martingale $\int_0^x 2\varphi(y)\sqrt{Z(y)}\,dB(y)$ in its Dubins–Schwarz (1965) form $w(A_x)$, where $(w(u), u \geq 0)$ is a Brownian motion, and $A_x = 4\int_0^x dy\varphi^2(y)Z_y$, we may represent Y_x as: $\gamma(A_x)$ where

[3] The condition at infinity is related to the requirement that $f^+(x)x$ integrates to infinity.

$$\gamma(u) = \frac{t}{l} - \frac{k}{4}u + w(u).$$

It follows that $\gamma(4\int_0^\infty dy(ky+l)^{-2}Z_y)$ is Y_∞, which by construction is zero. Hence,

$$4\int_0^{\sigma(t)} f^+(W(s))\,ds = 4\int_0^\infty dx(kx+l)^{-2}Z_x = T_0(\gamma),$$

the time at which the Brownian motion γ with drift $-k/4$ and initial value $\frac{t}{l}$ reaches zero. The density of $T_0(\gamma)$ is given by

$$P(T_0(\gamma) \in ds) = \frac{t/l}{\sqrt{2\pi s^3}} \exp\left(-\frac{(t/l)^2}{2s}\right) \exp\left(\frac{tk}{l} - \frac{k^2 s}{32}\right) ds.$$

The Laplace transform is then given by

$$E[\exp(-\lambda T_0(\gamma))] = \exp\left(-\frac{t}{l}\left(\sqrt{\frac{k^2}{16} + 2\lambda} - \frac{k}{4}\right)\right)$$

and the Lévy density is

$$k(x) = \frac{t}{l}\frac{\exp\left(\frac{-k^2}{32}x\right)}{\sqrt{2\pi x^3}}.$$

We observe in this example that the force function and the Lévy density are both inversely related to the parameters k and l. The density of the Lévy measure density is in the CGMY class of Lévy densities.

3.10. Example with Increasing Force Function

Suppose that

$$f^+(x) = \theta \exp(\alpha x), \qquad \alpha, \theta > 0.$$

For this case, following Jeanblanc, Pitman and Yor (1997, Ex. 6), we observe that the solution to the Sturm–Liouville equation with the boundary condition $\psi(0) = 1$ and $\psi(\infty) = 1$ is

$$\psi(x) = \frac{K_0\left(\frac{2\sqrt{2\lambda\theta}}{\alpha}\exp\left(\frac{\alpha}{2}x\right)\right)}{K_0\left(\frac{2\sqrt{2\lambda\theta}}{\alpha}\right)},$$

where K_0 is the modified Bessel function of the second kind of order zero. It follows on differentiation, noting that $K_1 = -K_0'$ and substituting into (3.30) that

$$E\left[\exp\left(-\lambda\int_0^{\sigma(t)} f^+(W(s))ds\right)\right] = \exp\left(-\frac{t}{2}\frac{\sqrt{2\lambda\theta}K_1\left(\frac{2\sqrt{2\lambda\theta}}{\alpha}\right)}{K_0\left(\frac{2\sqrt{2\lambda\theta}}{\alpha}\right)}\right).$$

For the Lévy measure associated with this Laplace transform we note from Donati-Martin and Yor (1997, p. 1055) that

$$\sqrt{\xi}\frac{K_1(\xi)}{K_0(\xi)} = \xi\int_0^\infty \exp(-\xi y)H_{-1}(y)\,dy = -\int_0^\infty (1-\exp(-\xi y))\frac{\partial}{\partial y}H_{-1}(y)\,dy.$$

Substituting $2\sqrt{2\lambda\theta}/\alpha$ for $\sqrt{\xi}$ and making the change of variable $8\theta y/\alpha^2 = z$ we obtain the Lévy density for this process as

$$k(x) = \frac{\alpha^3}{16\theta}\frac{1}{\pi^2}\int_0^\infty dz \exp\left(-z\frac{\alpha^2 x}{8\theta}\right)\frac{1}{J_0^2(\sqrt{z}) + Y_0^2(\sqrt{z})},$$

where we have differentiated H_{-1} using the definition provided in Donati-Martin and Yor (1997, equation (6.6)).

4. CONCLUSION

We argue in this paper that price processes, being semimartingales (an implication of the no-arbitrage condition) are time-changed Brownian motions. We observe that as time changes are increasing random processes they are for practical purposes purely discontinuous if they are not locally deterministic. This leads us to consider purely discontinuous models for the prices of financial assets. We show, by various examples, that one may generally relate this time change to a measure of price moves. Typically, for the homogeneous Lévy processes with completely monotone Lévy densities one measures time by an exponential weighting of the size of the price move.

The specific time changes considered in this paper are Poisson processes, gamma processes, general subordinators, and the inverse local time of Brownian motion at zero. In each case we exhibit the price process as the difference between two increasing processes, one recording the price increases and the other the price decreases. In each case we show how the price process may be viewed as Brownian motion evaluated at a random time that is related to the price moves.

We provide numerous examples of potentially empirically relevant Lévy processes with closed-form expressions for their Lévy densities and characteristic functions. The resulting models are tractable for both option pricing and statistical estimation.

APPENDIX

A.1. Derivation of Equation (3.20)

We may write the right hand side of (3.19) as

$$\begin{aligned}\alpha_1 N_d^2 - \alpha_2 N_s^2 &= \left(\sqrt{\alpha_1}N_d - \sqrt{\alpha_2}N_s\right)\left(\sqrt{\alpha_1}N_d + \sqrt{\alpha_2}N_s\right)\\ &= \sqrt{\alpha_1+\alpha_2}M\left(\sqrt{\alpha_1}N_d + \sqrt{\alpha_2}N_s\right),\end{aligned} \tag{A.1}$$

where $M = (\sqrt{\alpha_1}N_d - \sqrt{\alpha_2}N_s)/\sqrt{\alpha_1+\alpha_2}$ is another standard normal variate. Now project $\sqrt{\alpha_1}N_d + \sqrt{\alpha_2}N_s$ on M and write

$$\sqrt{\alpha_1}N_d + \sqrt{\alpha_2}N_s = \frac{\alpha_1-\alpha_2}{\sqrt{\alpha_1+\alpha_2}}M + \frac{2\sqrt{\alpha_1\alpha_2}}{\sqrt{\alpha_1+\alpha_2}}\widetilde{M}, \tag{A.2}$$

where $\widetilde{M}$ is a standard normal variate independent of M. Substitution of (A.2) into (A.1) shows that one may write

$$\alpha_1 N_d^2 - \alpha_2 N_s^2 = (\alpha_1-\alpha_2)M^2 + 2\sqrt{\alpha_1\alpha_2}|M|\widetilde{M}. \tag{A.3}$$

If we define by $\gamma_3(1/2)$, by the relation $M^2 = 2\gamma_3(1/2)$, one may then write (A.3) as

$$\alpha_1 N_d^2 - \alpha_2 N_s^2 = (\alpha_1 - \alpha_2)2\gamma_3(1/2) + 2\sqrt{\alpha_1\alpha_2}\sqrt{2\gamma_3(1/2)}\widetilde{M} \tag{A.4}$$

and division of (A.4) by 2, noting that (3.19) yields

$$\ln\left(\frac{p\left(\frac{1}{2\kappa}\right)}{p(0)}\right) = (\alpha_1 - \alpha_2)\gamma_3(1/2) + \sqrt{2\alpha_1\alpha_2}\sqrt{\gamma_3(1/2)}\widetilde{M} \tag{A.5}$$

or the result that the log price process is Brownian motion with drift $(\alpha_1 - \alpha_2)$ and volatility $\sqrt{2\alpha_1\alpha_2}$ evaluated at $\gamma_3(t)$.

We see from (A.5) that the price process is basically the difference of squares of normals. The nonnegative prevailing price buy and sell orders are essentially the squares of Gaussian variates, N_d, and N_s, respectively. The time change, M, is the square of the excess demand $(\sqrt{B} - \sqrt{S})^2$, which is the activity measure in this case.

A.2. Derivation of Equation (3.23)

First note that the characteristic function of $\ln(p(t)/p(0))$ may be written as

$$\phi_{\ln(p(t)/p(0))}(u) = \exp\left(-2t\int_0^\infty (1 - \cos(u\alpha x))k(x)\,dx\right).$$

On the other hand the characteristic function of $\sigma W(t) = Y(t)$ evaluated at a time change $\gamma_3(t)$ conditional on the time change is

$$\phi_{Y(\gamma_3(t))|\gamma_3(t)}(u) = \exp\left(-\frac{\sigma^2 u^2 \gamma_3(t)}{2}\right).$$

Suppose that $\gamma_3(t)$ is a Lévy process with Lévy measure $k_3(x)dx$, then the characteristic function of $Y(\gamma_3(t))$ is given by

$$\phi_{Y(\gamma_3(t))}(u) = \exp\left(-t\int_0^\infty \left(1 - e^{-(\sigma^2u^2/2)x}\right)k_3(x)\,dx\right).$$

For the log price process to be a time-changed Brownian motion, using the Lévy process $\gamma_3(t)$ for the time change, one must have

$$2\int_0^\infty (1-\cos(u\alpha x))k(x)\,dx = \int_0^\infty \left(1-e^{-(\sigma^2u^2/2)x}\right)k_3(x)\,dx. \tag{A.6}$$

We may let $\sigma^2/2=\alpha$, $\tilde{k}(y)=k(y/\alpha)$, $\tilde{k}_3(y)=k_3(2y/\sigma^2)$ and then write (A.6) as

$$2\int_0^\infty \left(1-\cos(ux)\right)\tilde{k}(x)\,dx = \int_0^\infty \left(1-e^{-u^2x}\right)\tilde{k}_3(x)\,dx. \tag{A.7}$$

Differentiating (A.7) with respect to u yields

$$\int_0^\infty \frac{\sin(ux)}{u}x\tilde{k}(x)\,dx = \int_0^\infty e^{-u^2x}x\tilde{k}_3(x)\,dx. \tag{A.8}$$

We now recall that

$$\frac{\sin(ux)}{u} = \frac{1}{2}\int_{-\infty}^\infty 1_{|y|<x}e^{iuy}\,dy \tag{A.9}$$

and

$$e^{-u^2x} = \int_{-\infty}^\infty \frac{1}{\sqrt{4\pi x}}e^{-y^2/4x}e^{iuy}\,dy. \tag{A.10}$$

Substituting (A.10) and (A.9) into (A.8) and using the uniqueness of Fourier transforms, we deduce that for each y

$$\int_0^\infty x\tilde{k}(x)\frac{1}{2}1_{|y|<x}\,dx = \int_0^\infty x\tilde{k}_3(x)\frac{1}{2\sqrt{\pi}}\frac{1}{\sqrt{x}}e^{-y^2/4x}\,dx. \tag{A.11}$$

Differentiating (A.11) with respect to $y\geq 0$ yields

$$\tilde{k}(y) = \frac{1}{2}\int_0^\infty \frac{\tilde{k}_3(x)}{\sqrt{\pi x}}e^{-y^2/4x}\,dx. \tag{A.12}$$

Equation (A.12) may be solved for $\tilde{k}_3(x)$ satisfying (3.23) when $\tilde{k}(y)$ is given as the Laplace equation (3.21). To observe this we recall that

$$e^{-ay} = \int_0^\infty \frac{a}{\sqrt{2\pi t^3}}e^{-y^2t/2}e^{-a^2/2t}\,dt. \tag{A.13}$$

Employing (A.13) we may write

$$\begin{aligned}\tilde{k}(y) &= \int e^{-ay}\rho(da)\\ &= \int \rho(da)\int_0^\infty \frac{dt}{\sqrt{2\pi t^3}}ae^{-y^2t/2}e^{-a^2/2t}.\end{aligned}$$

Making the change of variable $t=\frac{1}{2x}$ we obtain

$$\tilde{k}(y) = \int \rho(da)\int_0^\infty \frac{dx}{\sqrt{\pi x}}ae^{-y^2/4x}e^{-a^2x}.$$

It follows that defining $\tilde{k}_3$ by equation (3.23) satisfies (A.11) as was to be shown.

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