

Hotelling Rule

The problem is:

$$\begin{aligned}
 & \underset{Q(t), 0 \leq t \leq T}{\text{Max}} \int_0^T [P(t)Q(t) - C]e^{-rt} dt \\
 \text{s.t } S(0) &= S_0 \text{ given} \\
 \dot{S}(t) &= -Q(t).
 \end{aligned}$$

where

- $Q(t)$ is the rate of extraction;
- $S(t)$ is the stock of the resource;
- C are the constant extraction costs.

Remark 1 we assume that $S(T) = 0$.

Then,

$$H(Q, S, q, t) = [P(t)Q(t) - C]e^{-rt} - q(t)Q(t),$$

where $q(t)$ is the shadow price of the stock in present value (i.e., in moment 0).

The necessary conditions for an optimum are:

$$\begin{aligned}
 \frac{\partial H}{\partial Q} &= 0 \Leftrightarrow P(t)e^{-rt} = q(t) \\
 \dot{q} &= -\frac{\partial H}{\partial S_t} \Leftrightarrow \dot{q} = 0 \\
 \dot{S}(t) &= -Q(t), S(0) = S_0.
 \end{aligned}$$

Alternatively, we can consider H in current values:

$$H^C(Q, S, q, t) = P(t)Q(t) - C - \mu(t)Q(t),$$

where $\mu(t) = q(t)e^{rt}$ is the shadow price of the stock in moment t .

The necessary conditions for an optimum are:

$$\begin{aligned}
 \frac{\partial H^C}{\partial Q} &= 0 \Leftrightarrow P(t) = \mu(t) & (1) \\
 \dot{\mu} &= r\mu(t) - \frac{\partial H^C}{\partial S_t} \Leftrightarrow \dot{\mu} = r\mu(t) \\
 \dot{S}(t) &= -Q(t), S(0) = S_0.
 \end{aligned}$$

From (1) we have:

$$\frac{\dot{\mu}(t)}{\mu(t)} = r.$$

Since $P(t) = \mu(t)$,

$$\frac{\dot{P}(t)}{P(t)} = r,$$

the famous Hotelling rule. Integrating,

$$\begin{aligned} \int_0^t \frac{\dot{P}(s)}{P(s)} ds &= \int_0^t r ds \\ \Leftrightarrow [\ln P(s)]_0^t &= [rs]_0^t \\ \Leftrightarrow \ln P(t) - \ln P(0) &= rt \\ \Leftrightarrow \ln \frac{P(t)}{P(0)} &= rt \\ \Leftrightarrow P(t) &= P(0)e^{rt}, \end{aligned}$$

a different expression for the Hotelling rule.