Parameter estimation and exotic option pricing

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Parameter estimation and exotic option pricing

- Assume that the underlying asset follows the exponential of a particular Lévy process
- In order to estimate the parameters of the mode, we use market data, for example the prices of european calls on some index at some fixed date.
- We use present values to estimate the parameter and not past or historical data.

Parameter estimation and exotic option pricing

- As an example, in order to calibrate the pricing model on the data, we choose parameters of the Variance Gamma model in order to minimize the quadratic error between the market prices of the call options and the call options prices given by the model.
- After estimating the parameters, if we want to price path-dependent options or exotic options (for example, barrier options), we can use Monte Carlo techniques to simulate a large number of paths of the Variance Gammma process with the optimized parameters previously estimates and we can calculate the exotic options by Monte-Carlo method from the formula:
$$V(t) = e^{-r(T-t)}\mathbb{E}_Q\left[X|\mathcal{F}_t\right].$$

Model calibration

- Data for model calibration: for example the 77 call option prices on S&P 500 Index at the close of the market on 18 April 2002 (see [3], page 155)
- Characteristic function of the Variance-Gamma distribution with parameters (σ, ν, θ):

$$\Phi_{VG}\left(u; \sigma, \nu, \theta\right) = \left(1 - iu\theta\nu + \frac{1}{2}\sigma^2\nu u^2\right)^{-1/\nu}.$$

- We can define the Variance-gamma process as a Lévy Process $X_t^{(VG)}$ such that the distribution of the increment $X_{t+s} - X_s$ follows the Variance-Gamma law with parameters $\left(\sigma\sqrt{t}, \nu/t, t\theta\right)$ and

$$\mathbb{E}\left[e^{iuX_t^{(VG)}}\right] = \left(1 - iu\theta\nu + \frac{1}{2}\sigma^2\nu u^2\right)^{-t/\nu}.$$

Model calibration

In the following table, we can find 77 call option prices on the S&P 500 Index at the close of the market on 18 April 2002. On that day, the S&P 500 Index closed at 1124.47. We had values of $r = 1.9\%$ and $q = 1.2\%$ per year.

Strike	May 2002	June 2002	Sep. 2002	Dec. 2002	March 2003	June 2003	Dec. 2003
975			161.60	173.30			
995			144.80	157.00		182.10	
1025			120.10	133.10	146.50		
1050		84.50	100.70	114.80		143.00	171.40
1075		64.30	82.50	97.60			
1090	43.10						
1100	35.60		65.50	81.20	96.20	111.30	140.40
1110		39.50					
1120	22.90	33.50					
1125	20.20	30.70	51.00	66.90	81.70	97.00	
1130		28.00					
1135		25.60	45.50				
1140	13.30	23.20		58.90			
1150		19.10	38.10	53.90	68.30	83.30	112.80
1160		15.30					
1170		12.10					
1175		10.90	27.70	42.50	56.60		99.80
1200			19.60	33.00	46.10	60.90	
1225			13.20	24.90	36.90	49.80	
1250				18.30	29.30	41.20	66.90

Model calibration

- The Variance-Gamma process has the following properties:
  (1) no diffusion component and it is a pure-jump process.
  (2) it has infinite activity (infinitely many (small) jumps in any finite time interval)
  (3) it has paths of finite variation: $\int_{-1}^{1} |x|\, \nu_{VG}(dx) < \infty$.
- (4) it has Lévy measure

$$\nu_{VG}(dx) = \begin{cases} C \exp(Gx)\, |x|^{-1}\, dx & \text{if } x < 0, \\ C \exp(-Mx) x^{-1} dx & \text{if } x > 0, \end{cases}$$

where $C = 1/v > 0$, $G = \left(\sqrt{\frac{1}{4}\theta^2\nu^2 + \frac{1}{2}\sigma^2\nu} - \frac{1}{2}\theta\nu \right)^{-1} > 0$,

$M = \left(\sqrt{\frac{1}{4}\theta^2\nu^2 + \frac{1}{2}\sigma^2\nu} + \frac{1}{2}\theta\nu \right)^{-1} > 0.$

Model calibration

- Under the historical probability measure $\mathbb{P}$, assume that the price of the risly asset is

$$S_t = S_0 \exp\left(m_H t + X_t^{VG}(\sigma_H, \nu_H, \theta_H) + w_H t\right),$$

where by the subscripts H we mean that these parameters are the ones under the historical probability measure $\mathbb{P}$. The parameter w_H is chosen such that it cancels the drift of the process $X_t^{VG}(\sigma_H, \nu_H, \theta_H)$ and therefore

$$w_H = \frac{1}{\nu_H} \ln\left(1 - \theta_H \nu_H - \frac{\sigma_H^2 \nu_H}{2}\right)$$

and m_H is the expected rate of return under $\mathbb{P}$.

- In order to price, we choose to estimate the parameters not under $\mathbb{P}$ but under $\mathbb{Q}$ (the risk neutral (RN) measure or equivalent martingale measure). Under $\mathbb{Q}$, the price process is

$$S_t = S_0 \exp\left(rt + X_t^{VG}(\sigma_{RN}, \nu_{RN}, \theta_{RN}) + w_{RN} t\right).$$

The parameter w_{RN} is chosen such that the discounted price process $\widetilde{S}_t = e^{-rt} S_t$ is a $\mathbb{Q}$ martingale (mean correcting equivalent martingale measure) and this results in

$$w_{RN} = \frac{1}{\nu_{RN}} \ln\left(1 - \theta_{RN} \nu_{RN} - \frac{\sigma_{RN}^2 \nu_{RN}}{2}\right).$$

In practice, we need to calculate the characteristic function at the point $1/i$.

Algorithm

- For a set of market prices of N calls, we choose the risk neutral parameters such that the quadratic error between market prices and the prices given by the model of the call options is minimum, and is given by the root-mean-square error

$$RMSE = \min_{\sigma_{RN},\nu_{RN},\theta_{RN}} \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left[(\text{market price})_i - (\text{calculated price})_i\right]^2}.$$

- Calls prices are calculated by the Fourier Transform method with the FFT algorithm
- The grid of the logarithm of the strike is such that allows to interpolate with an acceptable error the prices of options for the strikes which are really traded on the market.

Some results

- We now present some results obtained by Schoutens and described in [3] (page 81), for the calibration procedure whith the CGMY model:

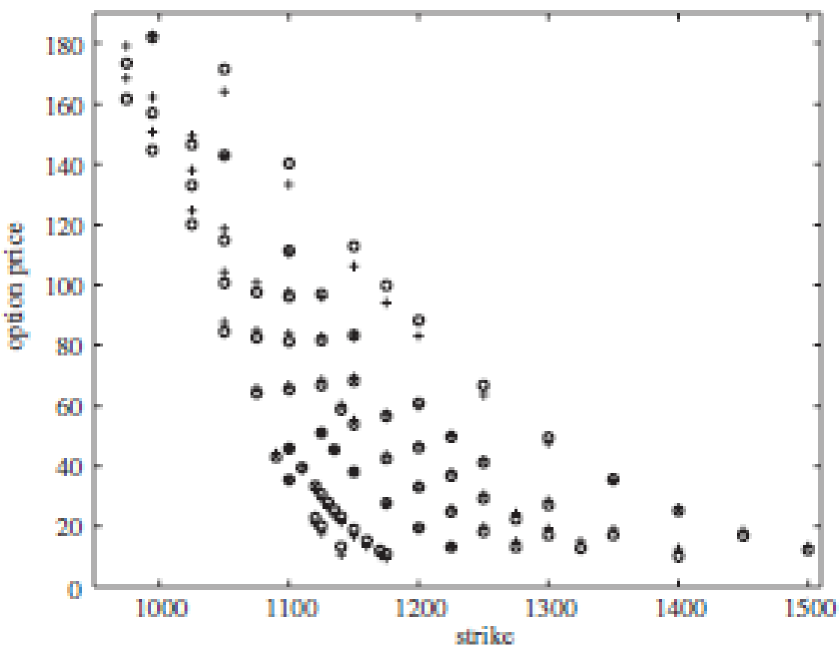

Figure 6.3 CGMY (mean-correcting) calibration of S&P 500 options (circles are market prices, pluses are model prices).

Comparison results

From [3], page 83,

Table 6.4 Lévy models (mean-correcting): APE, AAE, RMSE, ARPE.

Model	APE (%)	AAE	RMSE	ARPE (%)
BS	8.87	5.4868	6.7335	16.92
CGMY	3.38	2.0880	2.7560	4.96
GH	3.60	2.2282	2.8808	5.46
VG	4.67	2.8862	3.5600	7.56
NIG	3.97	2.4568	3.1119	6.17
Meixner	4.19	2.5911	3.2451	6.71

Pricing exotic options by the Monte-Carlo method

- The payoff of an "Up and In call option" (barrier option) with strike K and barrier H is equal to the payoff of the european call, if the underlying reached or crossed the barrier H between time 0 and T. If the barrier has not been reached, then the payoff is 0.
- After estimating the parameters, assume that we want to price an exotic option (for example, a barrier option of the type "Up and in"). We can use the Monte-Carlo method from the formula:

$$V(0) = e^{-rT}\mathbb{E}_Q\left[(S_T - K)^+ \mathbf{1}_{\{\max(S_t;0\leq t\leq T)\geq H\}}(\omega)\right],$$

  where H is the barrier level.
- Note that if $H \leq K$, the up and in call and the european call with strike K and maturity T have the same value, because if $S_T > K$ then $S_T > H$ also.

Monte Carlo algorithm

Monte Carlo algorithm:

- (1) We assume that the parameters of the risk neutral process were previously calibrated on the market prices of european calls by the method previously described
- (2) A large number N of trajectories of the risk neutral process is simulated on a regular time grid.
- (3) For each trajectory i ($i = 1, 2, ..., N$) we calculate the payoff of the option by formula:

$$C_i = \left[(S_T - K)^+ \mathbf{1}_{\{\max(S_t; 0 \le t \le T) \ge H\}} (\omega_i) \right]$$

- (4) The final price of the option can be estimated by the discounted mean of the payoff for the N trajectories:

$$\widehat{V}(0) = e^{-rT} \frac{1}{N} \sum_{i=1}^{N} C_i .$$

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Schoutens, W. (2002). Lévy Processes in Finance: Pricing Financial Derivatives. Wiley.